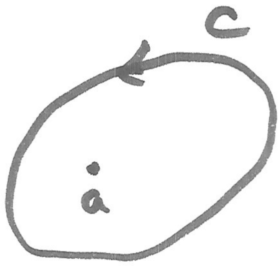
7.11.16

# Cauchy's Integral formulae & Related Theorem

- $f(z) \to$ analytic inside & on a simple closed curve $C$.

- $a \to$ any pt. inside $C$.




Then
$$\boxed{f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz.}$$

$$f^{n}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad n = 1, 2, 3, \ldots$$

- $f(z)$ analytic $\Rightarrow$ $f'(z)$ exists.

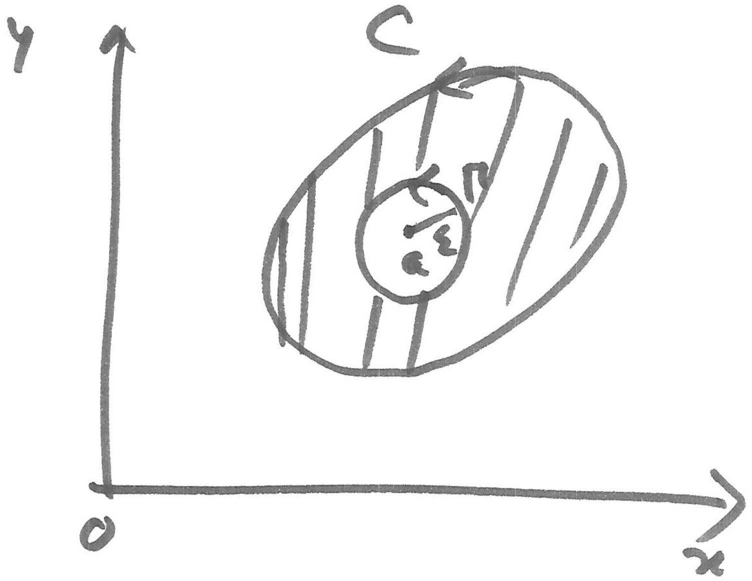
  & $f^{n}(z)$ exists $\forall n \geq 2$.
  +ve integer.

proof. ~~Sket~~ (Sketch).

$$\frac{f(z)}{z-a}$$

$\Downarrow$

analytic inside & on C except at $z=a$.




$\Gamma: |z-a| = \varepsilon$.

$\varepsilon$ arb. small. $\varepsilon > 0$.

$$\oint_C \frac{f(z)}{z-a}\,dz = \oint_{\Gamma: |z-a|=\varepsilon} \frac{f(z)}{z-a}\,dz.$$

$z-a = \varepsilon e^{i\theta}$

$0 \le \theta < 2\pi$

$dz = \varepsilon i e^{i\theta} d\theta$

$$= \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}} \cdot \cancel{\varepsilon} i \cancel{e^{i\theta}}\, d\theta$$

$$= i \cdot \int_0^{2\pi} f(a+\varepsilon e^{i\theta})\, d\theta$$

Letting $\varepsilon \to 0$

$$= i \cdot \int_0^{2\pi} f(a)\, d\theta = f(a)\, 2\pi i$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

Leibnitz rule for differentiation under the integral sign.

$$\frac{d}{da} f(a) = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a} \left[ \frac{f(z)}{z-a} \right] dz.$$

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz.$$

$$\vdots$$

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

Example:

$$I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\, dz.$$

$C$ is any simple closed contour.
& $z = 1, 2$ are pts. inside $C$.

Soln:

$$I = \oint_C \left[\sin \pi z^2 + \cos \pi z^2\right] \left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz$$

$$= \oint_C \frac{\overbrace{\sin \pi z^2 + \cos \pi z^2}^{f(z)}}{z - \underbrace{2}_{a}}\, dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\, dz$$

$$= 2\pi i \left[\sin \pi 2^2 + \cos \pi 2^2\right]$$

$$- 2\pi i \left[\sin \pi 1^2 + \cos \pi 1^2\right]$$

$$= 4\pi i$$

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

C


$f(z)$ analytic

**Example:**

$$\oint \frac{e^{2z}}{(z+1)^4}\,dz \qquad e^{2z} \rightarrow \text{analytic} \qquad f(z) = e^{2z}$$

$c: |z| = 3.$

$$f^{n}(a) = \frac{n!}{2\pi i}\oint \frac{f(z)}{(z-a)^{n+1}} \qquad c: |z| = 3.$$


$z=-1$ inside

$c: |z| = 3.$

$$= \frac{2\pi i}{3!} f^{(3)}(-1), \text{ when } f(z) = e^{2z}.$$

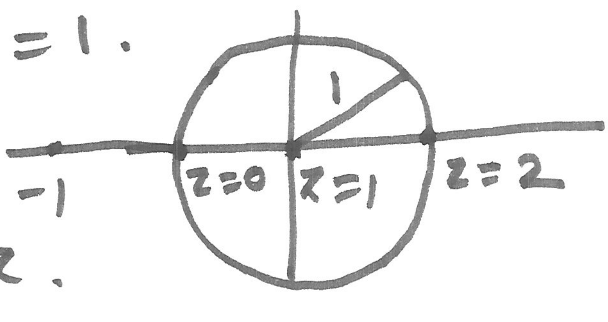
$$= \frac{2\pi i}{3!}\left(8e^{-2}\right).$$

**Example:**

$$I = \int \frac{3z^2+z}{z^2-1}\,dz \qquad z^2-1 \rightarrow (z-1)(z+1)$$

$c: |z-1| = 1.$

$$= \oint_c \frac{\overbrace{(3z^2+z)/(z+1)}^{f(z)}}{z-1}\,dz.$$

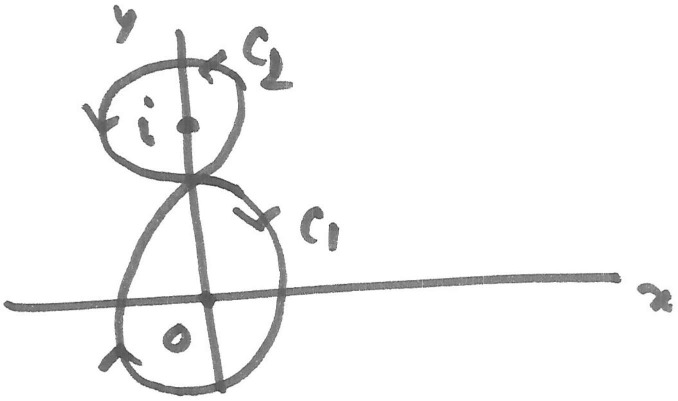


$$= 2\pi i \quad \frac{3.1^2+1}{1+1}$$ using Cauchy's Integral formula.

$$= 4\pi i$$

<u>Example</u>: Evaluate $\displaystyle\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$

where $C$ is the contour shown below.




<u>Soln.</u> $C$ is <u>not</u> <u>simple</u> closed contour

Think of $C$ as $C = C_1 \cup C_2$.

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$= -\oint_{C_1} \frac{z^3+3}{z(z-i)^2} + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz.$$

$C_1$ traversed in +ve sense

$C_2$ already +ve sense.

$z=0$ inside $C_1$ $\qquad$ $z=i$ inside $C_2$

$$= -2\pi i\left[\frac{z^3+3}{(z-i)^2}\right]_{z=0} + 2\pi i \cdot \left\{\frac{d}{dz}\left[\frac{z^3+3}{z}\right]\right\}_{z=i}$$

using $f^n(a) = \dfrac{n!}{2\pi i}\displaystyle\oint_C \dfrac{f(z)}{(z-a)^{n+1}}$

$$= 4\pi(-1+3i).$$

**Example:** Prove that

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{1.3.5\cdots(2n-1)}{2.4.6\cdots(2n)}, \text{ when } n=1,2,3,\ldots$$

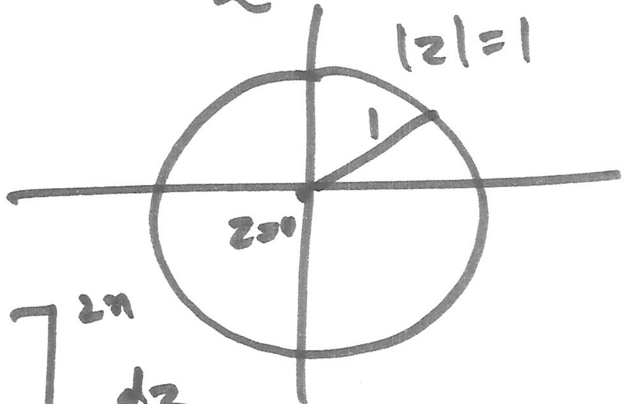
**Sol$^n$.**

let $\boxed{z=e^{i\theta}.}$, $0 \le \theta < 2\pi$.

Then $dz = \underbrace{e^{i\theta}}_{z}\, i\, d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$.

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$$

let $C: |z|=1$.
(unit circle)




$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \oint \left[\frac{z+\frac{1}{z}}{2}\right]^{2n} \frac{dz}{iz}$$

$C: |z|=1$.

$$= \frac{1}{2^{2n}.i} \oint_C \frac{1}{z}\left[z^{2n} + \binom{2n}{1} z^{2n-1}.\frac{1}{z} + \cdots \right.$$
$$\left. \cdots + \binom{2n}{k} z^{2n-k}\frac{1}{z^k} + \cdots + \frac{1}{z^{2n}}\right] dz$$

$$= \frac{1}{2^{2n} \cdot i} \oint \left[ z^{2n-1} + \binom{2n}{1} z^{2n-3} + \cdots + \binom{2n}{k} z^{2n-k-k-1} + \cdots + z^{2n-1} \right] dz$$

$C : |z| = 1$

$=$

$$\boxed{\oint_{C:|z|=1} \frac{dz}{z^n} = \begin{cases} 2\pi i & \text{if } n = 1 \\ 0 & \text{if } n > 1 \text{ +ve int.} \end{cases}}$$

$\downarrow$ proof using Cauchy's Int. formula.

$C$

$z = 0$

$$\oint_C \frac{\overset{=1}{f(z)}\, dz}{z^n}$$

$\downarrow$

$$\frac{2\pi i}{(n-1)!} f^{n-1}(0) = \begin{cases} 0 & \text{if } n > 1 \text{ +ve int} \\ 2\pi i & \text{if } n = 1 \end{cases}$$

$2n-2k-\cancel{1}=-\cancel{1} \Rightarrow k=n.$

$$= \frac{1}{2^{2n} \cdot i}\left[0+0+\cdots+\binom{2n}{n}2\pi i+0+0\cdots+0\right]$$

Simplify & get the result.

$$\oint z^{l}\,dz = 0,\ l>0$$

$C: |z|=1$

# Infinite Series - Taylor's & Laurent's Series.

## Power Series

- $\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$

  $\rightarrow$ power series in $z=a$.

- $z = a \rightarrow$ series convergent.

$$\sum_{n=0}^{\infty} u_n(z) = u_0(z) + u_1(z) + \cdots$$

$S_0(z) = u_0(z)$

$S_1(z) = u_0(z) + u_1(z)$

$\vdots$

$S_n(z) = u_0(z) + \cdots + u_n(z)$

$\{S_n(z)\}$

if $S_n(z) \rightarrow S(z)$ sum of the series

as $n \rightarrow \infty$.

then $\sum_{n=0}^{\infty} u_n(z)$ converges

o.w. diverges.

Example:

$$\sum_{n=1}^{\infty} \left(\frac{6n+1}{2n+5}\right)^n (z-2i)^n.$$

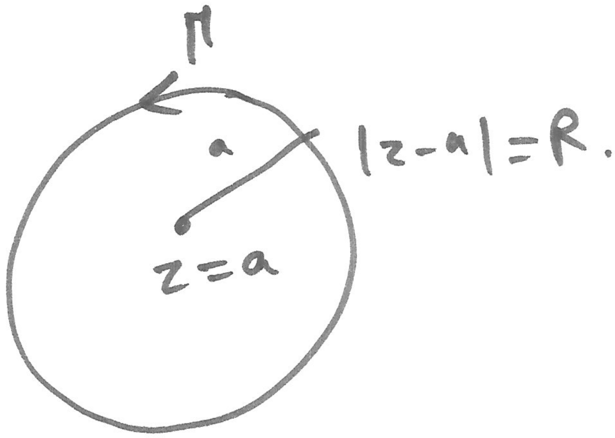
· It can be proved that $\exists$ a no. $\underline{R > 0}$

s.t. $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges for $\underline{|z-a| < R}$

& diverges for $|z-a| > R$, while

for $|z-a| = R$, it may or may not

converge.




Geometrically,

· inside $\Gamma : |z-a| = R$

$\rightarrow \sum_{n=0}^{\infty} a_n (z-a)^n$ converges.

outside $\Gamma$ diverges

on $\Gamma$ may or may not converge (undecided).

· $\Gamma \rightarrow$ circle of convergence

$R \rightarrow$ radius of convergence

· $R = \infty$ if $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges $\forall z$, $z$ finite

· $R = 0$ if $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges only at $z = a$.

## Determining R.

$$\sum_{k=0}^{\infty} a_k (z-a)^k$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \quad \text{or} \quad \lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

- Radius of convergence $\rightarrow$ $R = \dfrac{1}{L}$
- Circle of convergence $\rightarrow$ $|z-a| = R$.

**Example:**

$$\sum_{k=1}^{\infty} \frac{(-1)^k (z-1-i)^k}{k!}$$

$R = \infty$.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{(n+1)!} \, n! \right|$$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

**Example:**

$$\sum_{n=1}^{\infty} \left( \frac{6n+1}{2n+5} \right)^n (z-2i)^n$$

$R = ?$

- A power series converges uniformly & absolutely in any region which lies ~~on~~ inside its circle of convergence.

~~absolute~~

Absolute convergence of $\sum_{n=0}^{\infty} u_n(z)$ if

$\sum_{n=0}^{\infty} |u_n(z)|$ converges.

Absolute convergence $\sum |u_n(z)|$ $\Rightarrow$ ordinary convergence $\sum u_n(z)$

$\not\Leftarrow$

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ does not converge

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$= 1 - \frac{1}{2} + \frac{1}{3} - \dots$

alternating series, converge.

$\sum_{n=0}^{\infty} u_n(z)$ convergent, but ~~not~~ $\sum_{n=0}^{\infty} |u_n(z)|$ does not converge $\rightarrow$ $\sum_{n=0}^{\infty} u_n(z)$ conditionally convergent.

## Uniform convergence

$\sum_{n=0}^{\infty} u_n(z)$ converges uniformly if $\{S_n(z)\}$ converges uniformly.

$S_n(z)$: $n$th partial sum of $\sum_{n=0}^{\infty} u_n(z)$

i.e. $\forall\, \varepsilon > 0$ (arb. small),

we can find $N > 0$ (depending on $\varepsilon$, not on $z$).

s.t. $|S_n(z) - S(z)| < \varepsilon$ $\forall\, n > N$.

$\lim_{n\to\infty} S_n(z) = S(z)$.

- Power series - can be differentiated term by term.
  - can be integrated term by term.

- Sum of two power series
  → continuous in any region which lies entirely within its circle of convergence.

- Power series represents an analytic fn. within its circle of convergence

## Taylor's Series. expansion

of an analytic $f^n$.
$f(z)$ about a pt.

$\underline{\underline{z=a}}$

$$\sum_{n=0}^{\infty} a_n (z-a)^n$$

$$a_n = \frac{f^n(a)}{n!}$$

. $f(z)$ analytic at $z=a$.

Cauchy's integral formula


$$\frac{1}{\cancel{n!}} \, \frac{\cancel{n!}}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz.$$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \frac{f^n(z)}{n!} (z-a)^n \\ &= \sum_{n=0}^{\infty} a_n (z-a)^n \\ \text{where } a_n &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} \, dz. \end{aligned}}$$

• $z=a \rightarrow$ a singular pt. of $f(z)$.

then $f(z)$ cannot be expressed in powers of $z-a$ in Taylor's series or Maclaurin's series.

$\downarrow$

$a=0$.

Example: $f(z) = \dfrac{\sin z}{z^3}$.

$\downarrow$ not analytic at $z=0$.

$\downarrow$ no Maclaurin's series expansion.

Example: $e^z = 1 + z + \dfrac{z^2}{2!} + \cdots, \quad |z| < \infty$

$R = \infty$.

$\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \cdots \qquad R=\infty$

$\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \cdots \qquad R=\infty$

$\ln(1+z) = z - \dfrac{z^2}{2} + \dfrac{z^3}{3} - \cdots \qquad |z|<1$

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \cdots + (-1)^{n-1} \frac{z^{2n-1}}{2n-1} + \cdots$$

$|z| < 1.$

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots, \quad |z| < 1$$

when multiple-valued ($p$ fraction), result valid for that branch of the fn. which has value 1 at $z=0$.

<u>Example:</u> $f(z) = \dfrac{\sin z}{z^3}$, $\underline{z = 0.}$

not analytic

$$f(z) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

converges $\forall\, z$, except $z=0$.

i.e. converges $\forall z$, $|z| > 0$.

# Laurent's Theorem. (about an isolated singularity).

- $f(z) \to$ single-valued, analytic in $\boxed{D: \; R_2 < |z-a| < R_1}$
- $a \to$ isolated singularity of $f(z)$.

$C_1: |z-a| = R_1$

$C_2: |z-a| = R_2$


Then

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \quad \text{for } D: R_2 < |z-a| < R_1}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz, \quad n = 0, \pm 1, \pm 2, \ldots$$

Laurent's series expansion of $f(z)$ about an isolated singularity $z = a$.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \cdots$$

analytic part of the laurent's series.

principal part of the laurent's series.

- if the principal part is zero → Taylor's Series.

<u>Example:</u> Expand $f(z) = \frac{1}{1-z}$ in a Taylor's Series with center $z_0 = 2i$. find also the circle of convergence.

<u>Sol^n.</u> $R$ ↓ radius of convergence.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz .$$

$$\boxed{\text{distance from } z = z_0 \text{ of the nearest singularity of } f(z).}$$

$$f(z) = \frac{1}{1-z} = f(2i) + (z-2i)\frac{f'(2i)}{1!} + (z-2i)^2 \frac{f''(2i)}{2!} + \cdots$$

about $z = 2i$

singular pt. of $f(z)$ $\rightarrow$ $z = 1$, $z = 2i$

distance

$(1,0)$, $(0,2)$.

$$\sqrt{(1-0)^2 + (0-2)^2} = \sqrt{5} .$$

$R = \sqrt{5}$. (Check) using Root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} .$$

" Ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1/R$$