31.10.16

Singular points.

A pt. at which $f(z)$ is not analytic is called singular pts.

Classification.

1. Isolated singularities.

- $f(z)$ has isolated singularity at $z = z_0$ if $\exists$ a deleted δ-nbd ($\delta > 0$, arb. small) of z_0 containing no singularity.

$f(z)$, z_0.

$|z - z_0| = \delta$

no other singularity except z_0.

- $\boxed{0 < |z - z_0| < \delta.}$

$\downarrow$

deleted δ-nbd.

- if no such δ can be found, then z_0 is called a non-isolated singularity.

- if z_0 in not a singular pt. & we can find $\delta > 0$ s.t. $|z - z_0| = \delta$ encloses no singular pt, then z_0 is an ordinary pt.

$f(z)$ a fu^n.
z_0 a pt.

$\rightarrow$ z_0 ordinary pt

$\rightarrow$ z_0 is a singular pt
(where $f(z)$ fails to be analytic).

2. Poles.

- if we can find a +ve integer n

s.t. $\boxed{\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0,}$

then $z = z_0$ is a pole of order n.

Example: (i) $f(z) = \dfrac{1}{(z-2)^3}$. has a pole of order 3 at $z=2$.

(ii) $f(z) = \dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$

$z=1 \rightarrow$ pole of order 2.

· $n=1 \rightarrow$ simple pole.

Zero of order n

If $g(z) = (z-z_0)^n f(z)$ when $f(z_0) \neq 0$, $n > 0$, int.

Then $z = z_0$ is a zero of order n of the fn. $g(z)$.

- $z = z_0$ is a pole of order n of the fn. $\frac{1}{g(z)}$.

3. Removable Singularities

An isolated singular pt. z_0 is called removable singularity of $f(z)$ if

$$\cancel{f(z_0) =} \lim_{z \to z_0} f(z) \text{ exists.}$$

By defining $\boxed{f(z_0) = \lim_{z \to z_0} f(z)}$, it can be shown that $f(z)$ is not only continuous at z_0, but it is also analytic at z_0.

Example: $f(z) = \frac{\sin z}{z}$ → has removable singularity at $z=0$.

as $\lim_{z \to 0} f(z) = 1$.

4. Essential singularities.

An isolated singularity which is not a pole or a removable singularity is called an essential singularity.

$$\lim_{z \to 0} (z-z_0)^{n} f(z) = A \neq 0$$

$$e^{\frac{1}{z}} = \cancel{\frac{1}{z}} \cancel{1+\frac{1}{z}} \cancel{\frac{1}{1/z}} + \cancel{\frac{1}{(1/z)^2 \, 2!}} + \cancel{\frac{1}{(1/z)^3 \, 3!}} + \cdots$$

$$= 1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 \cdot \frac{1}{2!} + \cdots$$

- If a fn. has isolated singularity, then it is either a removable pole or essential singularity.

removable pole → non-essential singularity

e.g. $z = z_0$ is an essential singularity if $\exists$ no +ve int. n s.t.

$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$$

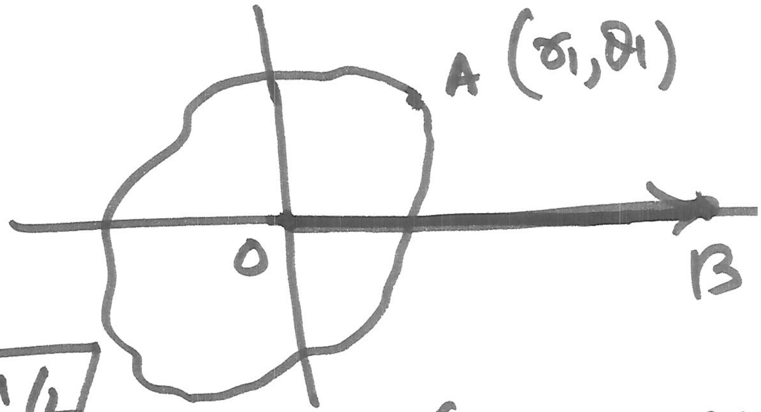
5. Singularities at infinity.

Type of singularity of $f(z)$ at $\underline{z = \infty}$.

$\downarrow$ same as

Type of singularity of $\underline{f\left(\frac{1}{w}\right)}$ at $\underline{w = 0}$.

6. Branch pts.



- ~~ ~~ $w = z^{1/2}$
- $z = r_1 e^{i\theta_1}$

$\Rightarrow \boxed{w = r_1^{1/2} e^{i\theta_1/2}}$ at A $(r = r_1, \theta = \theta_1)$

- After 1 complete circuit

$\theta = \theta_1 + 2\pi$.

so $w = r_1^{1/2} e^{i\frac{\theta_1 + 2\pi}{2}}$

i.e. $\boxed{w = -r_1^{1/2} e^{\theta_1/2}}$

$\underline{\hspace{4em}}$ different value of w

- After ~~another~~ a second circuit,

$\theta = \theta_1 + 4\pi$.

$\hookrightarrow$ $\boxed{\omega = r_1^{1/2} e^{i\theta_1/2}}$

$\rightarrow$ same value of ω with which we started.

- if $0 \leq \theta < 2\pi$, we are on one branch of the multiple-valued fn.
- if $2\pi \leq \theta < 4\pi$, we are on the other branch of the fn.
- each branch is single valued.

Branch line or branch cut (OB).

In order to keep the fn. single-valued, we set up an artificial barrier such as OB which we agree not to cross. This barrier is called a branch line or a branch cut.

- Branch point $\rightarrow$ the pt. O is called the branch point. (singular pt.).

$\boxed{w = z^{1/2}}$ algebraic fun.

$w^2 - z = 0 \longrightarrow$ poly. eqn. in w of degree 2.

$\rightarrow$ has two roots in $\mathbb{C}$.

Example:

i) $f(z) = \ln(z^2 + z - 2)$

$= \ln(z+2)(z-1)$.

has branch pts. at $z = -2, z = 1$

ii) $f(z) = (z-2)^{1/2}$

$z = 2$ is a branch pt.

Example: $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$.

check. $\underline{z = 0}$ is a branch pt. or not?

Soln.

$z = \underline{re^{i\theta}} = \underline{re^{i(\theta + 2\pi)}}$, $0 \leq \theta < 2\pi$

$f(re^{i\theta}) = \dfrac{\sin(r^{1/2} e^{i\theta/2})}{r^{1/2} e^{i\theta/2}}$ ✓

$$f\left(re^{i(\theta+2\pi)}\right) = \frac{\sin\sqrt{re^{i(\theta+2\pi)}}}{\sqrt{re^{i(\theta+2\pi)}}}$$

$$= \frac{\sin\left(-r^{\frac{1}{2}}e^{i\theta/2}\right)}{-r^{\frac{1}{2}}e^{i\theta/2}}.$$

$$= \frac{\sin\left(r^{1/2}e^{i\theta/2}\right)}{r^{1/2}e^{i\theta/2}}.$$

$\Rightarrow$ $f(z)$ has only one branch.

& $z=0$ is not a branch pt.

Q) Type of singularity of $f(z)$ at $z=0$?

Since $\lim\limits_{z\to 0} \dfrac{\sin\sqrt{z}}{\sqrt{z}} = 1$, it follows that $z=0$ is a removable singularity.

Example: $f(z) = \sec\left(\frac{1}{z}\right)$.

Find the singularities of $f(z)$ & classify them.

Solⁿ. • Singularity occurs when $\cos\frac{1}{z} = 0$.

$$\Rightarrow z = \frac{2}{(2n+1)\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$$

• $z = 0$ itself a singularity.

essential singularity.

simple poles.

Complex integration and Cauchy's Theorem

Complex line integral.

$C \rightarrow$ finite length curve, $f(z)$ continuous at every pt. on C.

curves

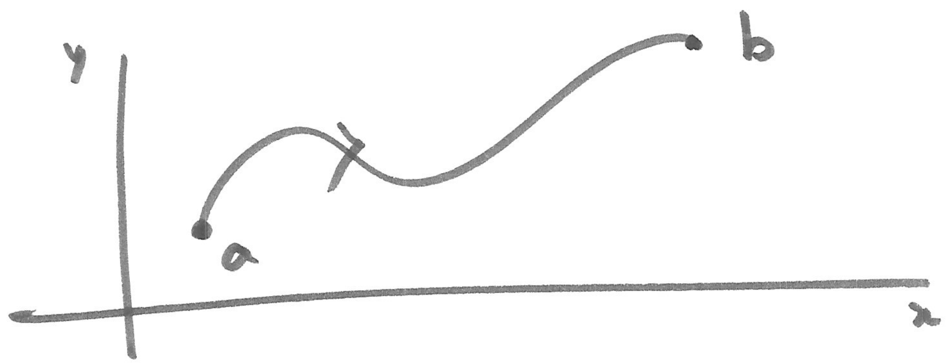
$x = \phi(t)$
$y = \psi(t)$

$z = x + iy$, $t_1 \le t \le t_2$

$= \phi(t) + i\psi(t) = z(t)$

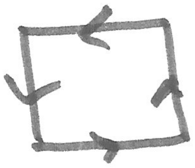
$\phi(t), \psi(t) \rightarrow$ real fn. of real variable t.

- $z \rightarrow$ continuous curve if $\phi(t), \psi(t)$ are continuous.



$z(t_1) = a$
$z(t_2) = b$.

- Closed curve $\rightarrow z(t_1) = z(t_2)$
- pairwise smooth curve / Contour.



$$\int_C f(z)\,dz \quad \text{or} \quad \oint_C f(z)\,dz$$

$\rightarrow$ line integral or contour integrals.

- $f(z) = u(x,y) + i\,v(x,y) = u + iv.$

$z = x + iy.$

$dz = dx + i\,dy.$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy).$$

$$= \int_C u\,dx - v\,dy + i \int v\,dx + u\,dy.$$

Example: $\int_C \bar{z}\, dz$ from $z=0$ to $z=4+2i$

i) along the curve $c: z = t^2 + it$.

ii) along the ~~curve~~ line from

$z=0$ to $z=2i$

& then the line

$z=2i$ to $z=4+2i$

Soln.

i)

$\int \bar{z}\, dz$

$C: \boxed{z = t^2 + it}$

from $z=0$ to $z=4+2i$

$\quad\quad\quad\downarrow \quad\quad\quad \downarrow$

$\quad\quad\; t=0 \quad\quad t=2$

$= \int_0^2 (t^2 - it)(2t+i)\, dt$

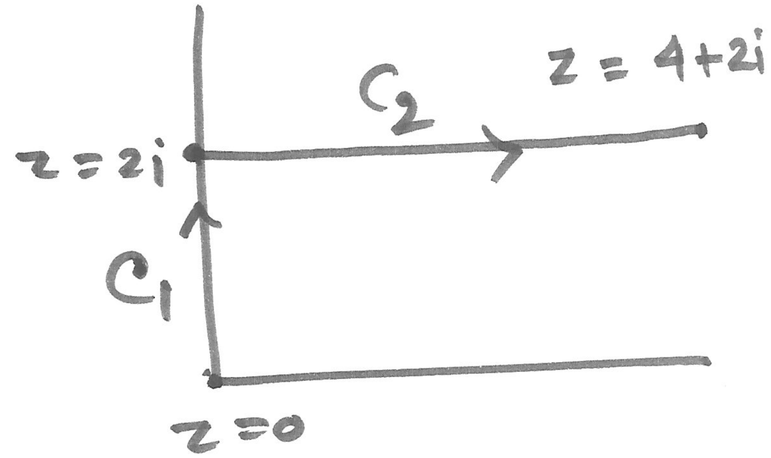
$z = t^2 + it$.

$dz = (2t+i)dt$

$= 10 - \frac{8i}{3}$ (Check).

(ii) $\displaystyle\int \bar{z}\, dz$

$C = C_1 \cup C_2$



$$= \int_C (x - iy)(dx + i dy) \qquad \begin{aligned} z &= x+iy \\ dz &= dx + i dy \end{aligned}$$

$$= \int_{C = C_1 \cup C_2} x\,dx + y\,dy + i \int_{C = C_1 \cup C_2} x\,dy - y\,dx.$$

$x = t^2, \; y = t.$

- along C_1, $\quad x = 0,\ dx = 0$

  y from 0 to 2.

$$\int_0^2 (0 \cdot 0 + y\,dy) + i \int_0^2 (0\,dy - y(0))$$

$$= \left.\frac{y^2}{2}\right]_0^2 = \frac{4}{2} = 2.$$

- along C_2, $\quad y = 2,\ dy = 0.$

  x from 0 to 4.

$$\therefore \int_{C_2} x\,dx + y\,dy + i \int_{C_2} x\,dy - y\,dx$$

$$= \int_0^4 \left(x\,dx + 2.0\right) + i \int_0^4 \left(x(0) - 2\,dx\right)$$

$$= \left.\frac{x^2}{2}\right]_0^4 + i\left[-2x\right]_0^4$$

$$= \frac{16}{2} + i(-2)4.$$

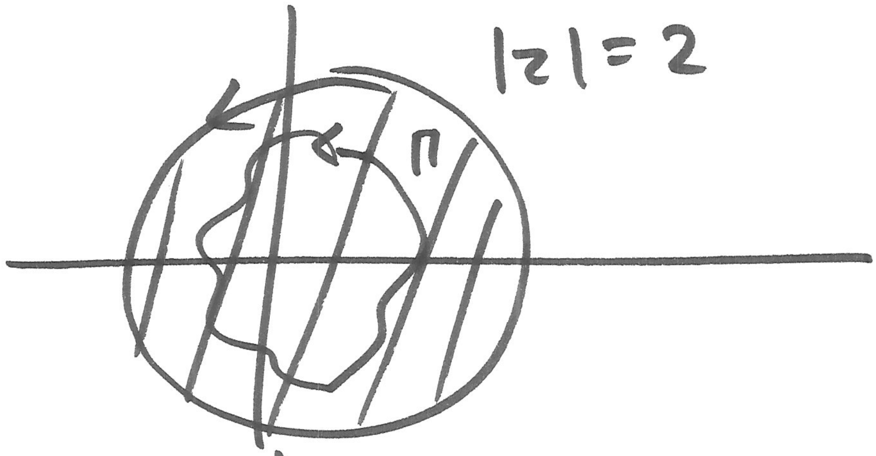
$$= 8 - 8i$$

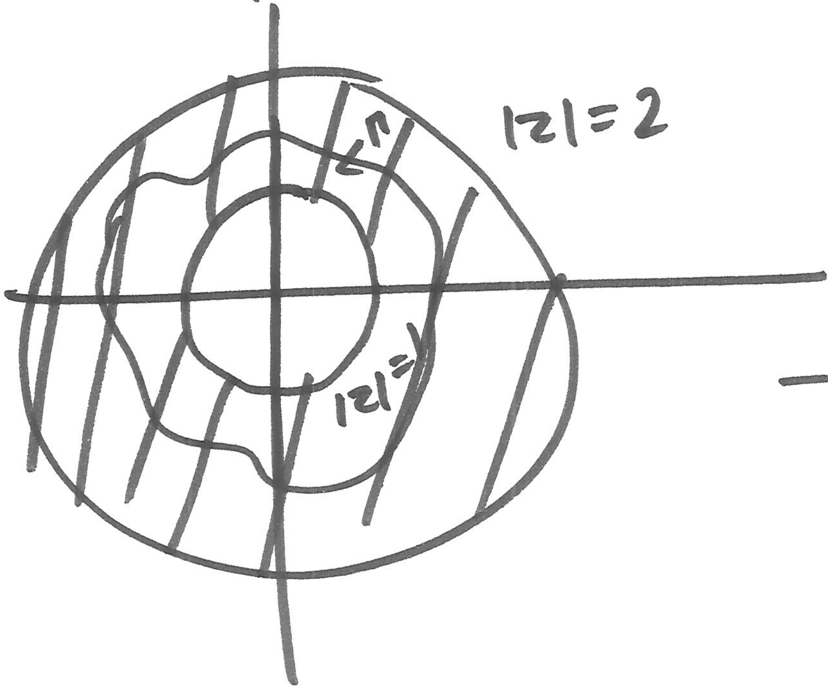
$$\int_C \bar{z}\,dz = 2 + 8 - 8i = 10 - 8i$$

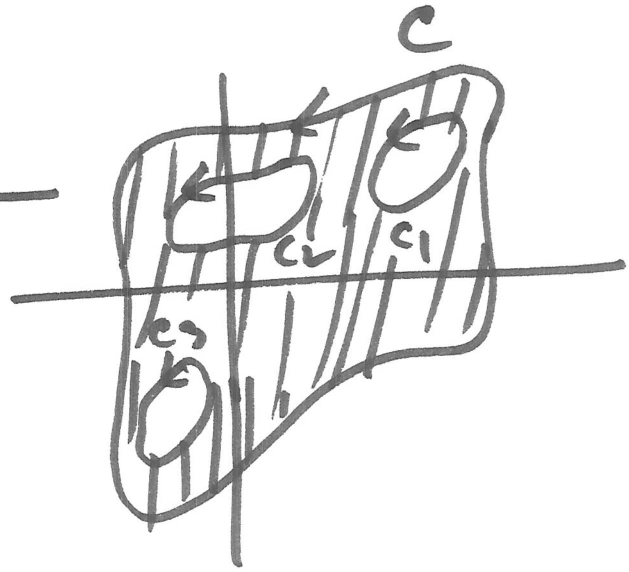
$$\int_C f(z)\,dz$$

if analytic, line integral is independent of path. (will be shown)

Simply connected domain / multiply connected domain.







. Any curve Γ inside the region R

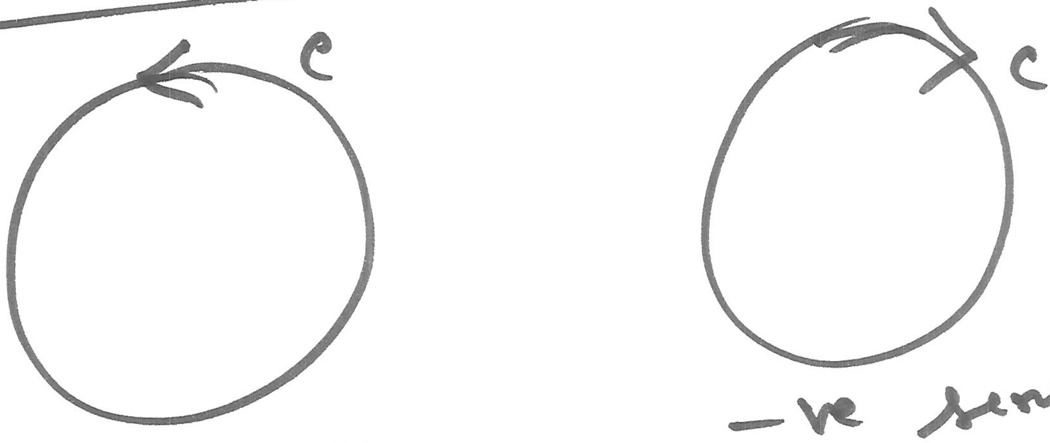
→ if Γ can be shrunked to a pt. without leaving R.

→ R simply connected region

. Otherwise → multiply connected region.

→ holes.

Convention regarding traversal of a closed path.



+ve sense (counter clockwise).

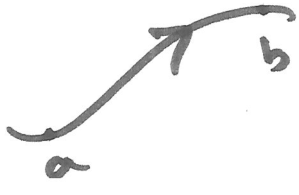
−ve sense (clockwise)

- $$\int_C f(z)\,dz.$$

- $$\int_C \big(f(z)+g(z)\big)\,dz = \int_C f(z)\,dz + \int g(z)\,dz$$

- $$\int_C A f(z)\,dz = A \int f(z)\,dz$$

- $$\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz.$$



$$\oint_a^b$$

Cauchy's Integral Theorem.

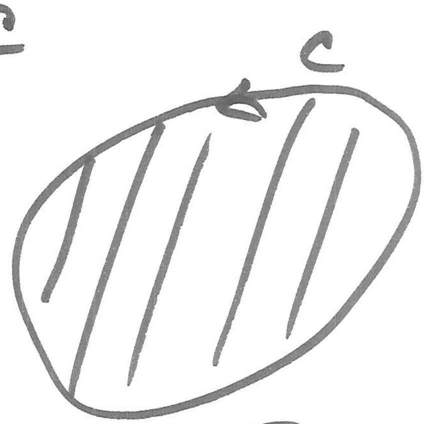
(Cauchy's Th.).

- $f(z) \rightarrow \boxed{\text{analytic}}$ in a closed region D

  → simple connected

  → multiply connected

& on its boundary C → closed curve.



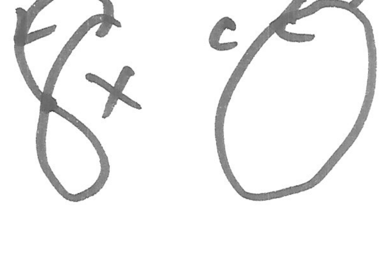
Then

$$\boxed{\oint_C \underline{f(z)}\,dz = 0.}$$

Jordan curve Th.

- C → continuous closed curve that does not intersect itself.

→ Jordan curves.



- C divides the plane into two regions, inside & outside of the curve resp'y.

Converse of Cauchy's Theorem.

(Morera's Theorem).

- $f(z) \rightarrow$ Continuous in a simply-connected domain D

and

- $\oint_C f(z)dz = 0$ around every simple closed curve C in D.

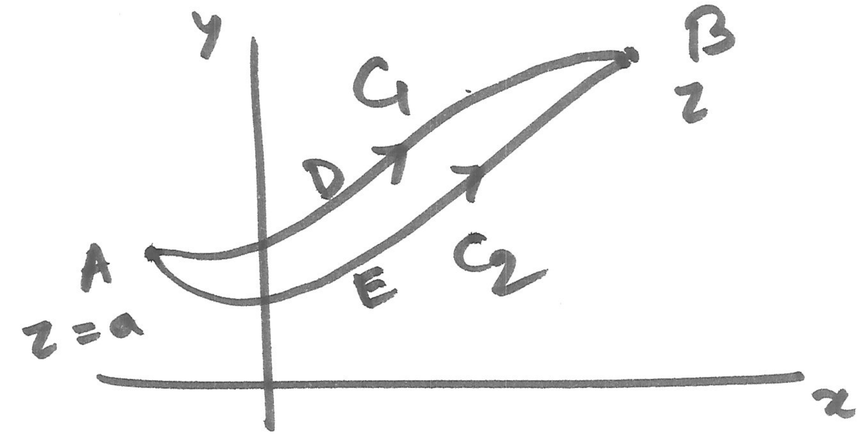
Then $f(z)$ is analytic in D.

Some consequences of Cauchy's Theorem

- Let $f(z)$ be analytic in a simply connected domain D.

Theorem. If a and z are any two pts. in D, then $\int_a^z f(z)dz$ is independent of path in D joining a and z.

proof.



By Cauchy's Theorem,

$$\int_{ADBEA} f(z)\,dz = 0 .$$

analytic

i.e. $$\int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0 .$$

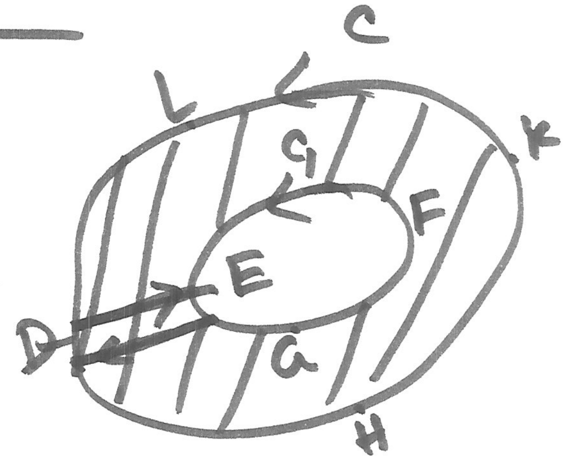
$ADB \to C_1$

i.e. $$\int_{C_1} f(z)\,dz = -\int_{BEA} f(z)\,dz .$$

$$= \int_{AEB} f(z)\,dz$$

$$= \int_{C_2} f(z)\,dz .$$

Deformation of contours.



- $f(z)$ analytic fun^n. in the region between C & C_1

Then
$$\boxed{\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.}$$

value of line integral over a complicated contour C → value of the line integral over a more convenient contour C_1 lying inside C.

Proof.

Take a cross cut DE.

By Cauchy's Th.,

$$\oint f(z)\,dz = 0$$

Γ: DEFGD HKLD

i.e., $\cancel{\int_{DE} f(z)\,dz} - \oint_{C_1} f(z)\,dz + \cancel{\int_{ED} f(z)\,dz}$

$$+ \oint_C f(z)\,dz = 0.$$

$$\Rightarrow \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz.$$

<u>Generalization.</u>

$C_1, C_2, \ldots, C_n$

→ non-overlapping curves

→ entirely within C

$f(z)$ → analytic

Then $$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

Example: Evaluate $\oint_C \frac{dz}{z-a}$

when C is any simple closed curve and $\underline{z=a}$ is

(i) outside C ✓

(ii) inside C.

(i) $\oint_C \frac{dz}{z-a} = 0.$

By Cauchy's Integral Th.

as $z=a$ is outside C, $f(z)=\frac{1}{z-a}$ is analytic inside & on C.

(ii) $z = a$ inside C

$$\oint_C \frac{dz}{z-a} = \oint_\Gamma \frac{dz}{z-a}.$$

$\Gamma : |z-a| = \varepsilon$

$\Gamma : |z-a| = \varepsilon$, ε arb. small

$$|z-a| = \varepsilon$$

$$\Rightarrow z-a = \varepsilon e^{i\theta}, \quad 0 \le \theta < 2\pi$$

$$\boxed{z = a + \varepsilon e^{i\theta}.}$$

$$dz = \varepsilon i e^{i\theta} d\theta$$

$$= \int_0^{2\pi} \frac{\cancel{\varepsilon} i \cancel{e^{i\theta}} d\theta}{\cancel{\varepsilon} \cancel{e^{i\theta}}}$$

$$= i\theta \Big]_0^{2\pi}$$

$$= 2\pi i$$