24.10.16

Analytic fn.

$N_\delta(z_0)$

- $f(z)$ analytic at $z = z_0$ if $f(z)$ is differentiable at ~~in~~ every pt. z in $N_\delta(z_0)$.

$\to$ $f'(z)$ exists

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

-

- Necessary condition for a fn. to be analytic.

$f(z) = u(x,y) + i\,v(x,y)$, $z = x + iy$.

$\to$ $f(z)$ must satisfy CR - eqn. (Cauchy- Riemann).

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$

- $f'(z) = \lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$

$$= \begin{cases} u_x + i v_x & \text{Considering } h \text{ purely real.} \\ & z = x+iy \\ & f(z) = u(x,y) + iv(x,y) \\ & z+h = \underline{\underline{x+h}} + iy \\ & f(z+h) = u(x+h, y) + i\,v(x+h, y) \\ -iu_y + v_y & \text{Considering } h \text{ purely imaginary} \\ & \text{let } h = ik,\ k \text{ real} \\ & z+h = x + i(y+k) \\ & f(z+h) = u(x, y+k) + iv(x, y+k) \end{cases}$$

equating real part & imaginary part, we get

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$$ CR equ^n.

<u>Theorem.</u> For a complex fn. $f(z) = u(x,y) + iv(x,y)$ defined over a domain D to be differentiable at $z = x+iy$, it is <u>necessary</u> that u_x, u_y, v_x, v_y should exist & satisfy the Cauchy-Riemann differential equ^{ns}.

<u>Observation.</u> Conditions of the above theorem are not sufficient.

Example:

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0. \\ 0, & z = 0. \end{cases}$$

$u(x,y) + i v(x,y)$

$z \neq 0$

$$u(x,y) = \frac{x^3 - y^3}{x^2+y^2}, \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

At: $z = 0$ or $(0,0)$.

Claim. 1 CR equ^{n}s are satisfied at $(0,0)$.

$$u_x(0,0) = v_y(0,0)$$
$$u_y(0,0) = -v_x(0,0)$$

Claim 2 $f'(z)$ does not exist at $z=0$.

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3 - 0^3}{h^2+0^2} - 0}{h} = 1.$$

(check)

$u_y(0,0) = -1$

$v_x(0,0) = 1 = v_y(0,0)$

$\Rightarrow$ CR equ^{ns} are satisfied at $(0,0)$.

– proves claim 1.

(b) $$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$\cancel{= \lim_{(x,y)\to(0,0)} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}}{x+iy}$$

$h = x+iy$

$h \to 0 \Rightarrow (x,y)\to(0,0)$

$$= \begin{cases} -(1-i) & \text{along } x=0. \to \dfrac{-\cancel{y^3}(1-i)}{i\,\cancel{y^3}} = -(-i+ \\ 1+i & \text{along } y=0. \end{cases}$$

$\Rightarrow$ $f'(0)$ does not exist; establishing claim 2

Sufficient conditions for a f^n. $f(z)$ to be analytic

- $f(z) = u(x,y) + iv(x,y)$
  $\rightarrow$ continuous single-valued f^n.
- $z = x+iy \in D$, D is the domain of $f(z)$.
- $u_x, u_y, v_x, v_y \rightarrow$ all continuous & satisfy CR eqn.

$\Rightarrow$ Then $f(z)$ is analytic at $z = x+iy$

- Observation.

- $\omega = f(z) = u(x,y) + i\,v(x,y)$.
- $z = x+iy$, $\bar{z} = x - iy$.
- $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{z-\bar{z}}{2i}$

$u(x,y)$ & $v(x,y)$ can be regarded as f^n's of z & $\bar{z}$.

- if u_x, u_y, v_x, v_y exist & all continuous, then the condition that ω is independent of $\bar{z}$ is

$$\boxed{\frac{\partial \omega}{\partial \bar{z}} = 0.}$$

$\omega \rightarrow f^n$. z & $\bar{z}$

i.e. $\frac{\partial}{\partial \bar{z}}(u+iv)=0$, $\quad \omega = f(z) = u+iv$.

x, y

$z, \bar{z}$

$$\Rightarrow \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) = 0.$$

$$\Rightarrow u_x\cdot\frac{1}{2} + u_y\left(-\frac{1}{2i}\right) + i\left[v_x\cdot\frac{1}{2} + v_y\cdot\left(-\frac{1}{2i}\right)\right] = 0.$$

or $(u_x - v_y) + i(u_y + v_x) = 0 + i0$.

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}$$

$$\Rightarrow \boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}} \rightarrow \text{CR eqn}^{\text{ns}}!$$

- if $f(z)$ analytic fn. of z, then x & y can occur only in the combination of $x+iy$.
- $f(z) = u(x,y) + i\,v(x,y) \longrightarrow \boxed{u(z,0) + i v(z,0)}$.

· let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be analytic.

Then $u(x,y)$, $v(x,y)$ $\rightarrow$ Conjugate fn^{ns}.

· Harmonic fn^{n}!

if $\nabla^2\phi = 0$
$\Rightarrow$ ϕ is harmonic.

i.e. $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0.$

Theorem. (The real & imaginary parts of an analytic fn^n. ~~satisfy~~ laplace's diff. equ^n.

(Assume that 2^{nd}. order partial derivatives of u & v exist & are continuous fn^{ns} of x,y).

$f(z) = u(x,y) + iv(x,y)$
$z = x+iy$.
analytic
$\Rightarrow$ ~~$\nabla^2 u = 0$~~
&
$\nabla^2 u = 0$, $\nabla^2 v = 0$.
when $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

proof. f analytic

$\Rightarrow$ $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$

claim $u_{xx} + u_{yy} = 0$.

$\Rightarrow u_{xx} = v_{xy} = v_{yx} = -u_{yy}$

$\Rightarrow u_{xx} + u_{yy} = 0.$

- $f(z)$ analytic $\Rightarrow u, v$ both harmonic fn's

$\Downarrow$

Conjugate harmonic fn!

__Theorem.__ If the harmonic fn. u & v satisfy CR-eqn., then $u+iv$ is an analytic fn.

$\rightarrow$ used to solve problems.

__Example:__ (Applications of CR eqn).

i) Prove that $\boxed{u(x,y) = e^{-x}(x\sin y - y\cos y)}$ is harmonic.

ii) find $v(x,y)$ s.t. $f(z) = u+iv$ is analytic

iii) find $f(z)$ in terms of z.

Sol^n.

(i) check by yourselves

$u_{xx} + u_{yy} = 0.$

ii) form CR eqn.

① — $u_x = v_y = -e^{-x}(x\sin y - y\cos y) + e^{-x}\sin y$

② — $u_y = -v_x = e^{-x}(x\cos y + y\sin y - \cos y).$

Integrate ①;

$$v = -e^{-x}\left(-x\cos y - y\sin y + \cancel{\int \sin y}\right) \quad \text{(}\int \sin y = -\cos y\text{)}$$

$$- e^{-x}\cancel{\cos y} + F(x),$$

($F(x)$ is a fn. of x).

③ — $v = y e^{-x}\sin y + x e^{-x}\cos y + F(x).$

②, ③ $\Rightarrow$

$$-y\cancel{e^{-x}}\sin y + e^{-x}\cancel{\cos y} - x\cancel{e^{-x}}\cos y + F'(x)$$

$$= -e^{-x}(x\cancel{\cos y} + y\cancel{\sin y} - \cancel{\cos y}).$$

$\Rightarrow F'(x) = 0. \Rightarrow F(x) = c$, a const.

$$\boxed{v = y e^{-x}\sin y + x e^{-x}\cos y + c}$$

(iii) $f(z) = u + iv. = u(z, 0) + iv(z, 0).$

$\downarrow$ $e^{-z}(0 - 0).$

$= i z e^{-z} + ic.$

$0 + z e^{-z} + c.$

$\omega = f(z) = u(x, y) + iv(x, y), \quad z = x + iy,$

$\bar{z} = x - iy.$

$$\boxed{x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}}$$

$$f(z) = u\left[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right] + iv\left[\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right].$$

$\rightarrow$ identity in two independent variables z & $\bar{z}$.

$\boxed{\text{put } z = \bar{z}} \Rightarrow \boxed{f(z) = u[z, 0] + iv[z, 0]}$

putting $y = 0$.

- $f'(z) = u_x + i v_x$ $\qquad u_x = v_y$, $u_y = -v_x$.

$= u_x - i u_y$

Milne - Thompson method to construct regular fn.

Suppose u is known.

Let $u_x = \phi_1(x,y)$, $u_y = \phi_2(x,y)$.

Then $f'(z) = \phi_1(x,y) - i\phi_2(x,y)$

$= \phi_1(z,0) - i\phi_2(z,0)$.

Integrating,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c,$$

c is an arb. const.

- Similarly, if $v(x,y)$ is given, then

$$f'(z) = v_y + i v_x = \psi_1(x,y) + i\psi_2(x,y)$$
$$= \psi_1(z,0) + i\psi_2(z,0).$$

$$\Rightarrow f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c'$$

$c' \to$ arb. const.

when $v_y = \psi_1(z,0)$, $v_x = \psi_2(z,0)$.

<u>Applications.</u> (Milne-Thompson)

<u>Example:</u> Find the analytic fn. of which the <u>real part</u> is

$$e^{-x}\left\{(x^2-y^2)\cos y + 2xy\sin y\right\}.$$

<u>Soln.</u> $u(x,y) = e^{-x}\left\{(x^2-y^2)\cos y + 2xy\sin y\right\}.$

$$u_x = \phi_1(x,y) = e^{-x}\left\{2x\cos y + 2y\sin y\right\}$$
$$- e^{-x}\left\{(x^2-y^2)\cos y + 2xy\sin y\right\}$$

$$u_y = \phi_2(x,y) = \ldots$$

$$\therefore f'(z) = \phi_1(z,0) - i\phi_2(z,0).$$
$$= e^{-z}(2z - z^2).$$

$\therefore f(z) = \int e^{-z}(2z - z^2)\, dz + c.$

$= c + z^2 e^{-z}.$

$\frac{d}{dz}(z^3)$
$= 3z^2.$

Example: If $\boxed{u = e^x(x\cos y - y\sin y)}$, find the analytic fn. $u + iv$.

Soln.

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy.$$
$$= v_x\, dx + v_y\, dy.$$
$$= -u_y\, dx + u_x\, dy.$$

CR

$u_x = v_y$
$u_y = -v_x$

exact differential

$$= -e^x(-x\sin y - \sin y - y\cos y)\, dx$$
$$+ e^x(x\cos y - y\sin y + \cos y)\, dy.$$

$$\therefore\ v = \int e^{x}\left(x\sin y + \sin y + y\cos y\right)dx$$

$$+ \int \left(\text{those terms which do not contain } x\right) dy.$$

$$= \sin y\left(xe^{x} - e^{x}\right) + e^{x}\sin y + e^{x}y\cos y + 0 + c,$$

c const.

- $f(z) = u + iv$.

  → express it in terms of z.

  use $u(z,0) + i\,v(z,0)$

  or

  otherway.

- <u>Analytic fn.</u> $f(z) = u + iv$.

  → u, v both harmonic.

  → CR equn. satisfied

  → if u given, v can be constructed.

  & vice versa.

  → orthogonal system of curves

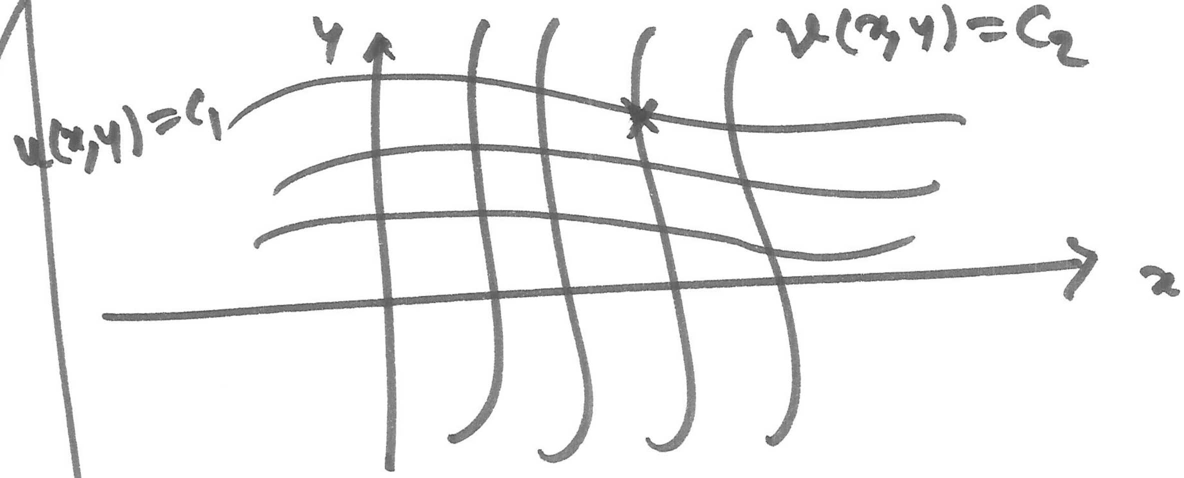
  $u(x,y) = c_1,\ v(x,y) = c_2$.

Orthogonal system of curves

- Two families of curves

$$u(x,y) = c_1, \quad v(x,y) = c_2.$$

Geometrically,



are said to be an <u>orthogonal system</u> of curves if they intersect at right angles at each of their pts. of intersection.

- Two such ~~curves~~ families of curves intersect orthogonally if

$$\boxed{u_x v_x + u_y v_y = 0.}$$

proof.

$$u(x,y) = c_1$$

$$u_x + v_y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{u_x}{u_y} = m_1.$$

$$v(x,y) = c_2 \Rightarrow \frac{dy}{dx} = -\frac{v_x}{v_y} = m_2.$$

$$m_1 m_2 = -1 \Rightarrow \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = -1.$$

$$u_x v_x + u_y v_y = 0.$$

- if $w = f(z) = u + iv$ is an analytic fu^n. of $z = x + iy$, then the curves

$$\left.\begin{array}{l} u(x,y) = \text{const} \\ v(x,y) = \text{const.} \end{array}\right\} \text{ on the } z\text{-plane}$$

intersect at right angles. $(f'(z) \neq 0)$.

$$f'(z) = \begin{cases} u_x + i v_x \\ -i u_y + v_y. \end{cases}$$

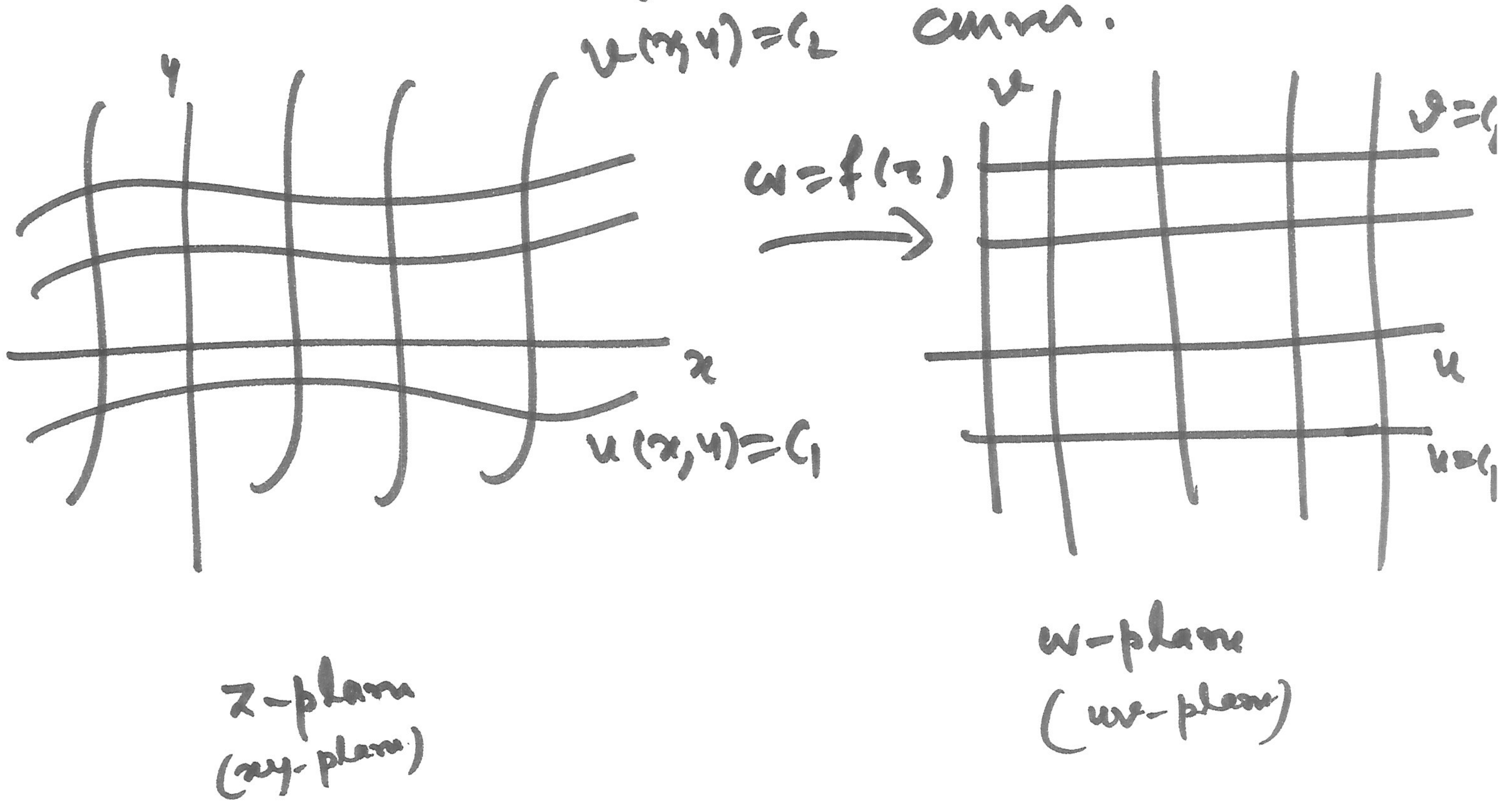
proof. $w = f(z)$ analytic $= u + iv$

$\Rightarrow$ CR eq^{ns}. are satisfied.

$$u_x = v_y, \quad u_y = -v_x.$$

Now $u_x v_x + \underline{\underline{u_y v_y}} = u_x v_x + (-v_x) u_x = 0.$

$\Rightarrow$ $u = \text{const}$, $v = \text{const}$. are orthogonal system of curves.



· $w=f(z)$ analytic, $f'(z) \neq 0$.

$C_1, C_2 \xrightarrow{f} C_1', C_2'$ (image curves)

in z-plane, in w-plane.

if angle between C_1, C_2 in z-plane = angle between C_1', C_2' in w-plane

then f is said to be a <u>conformal mapping</u>.

Example: analytic fn. $f = u + iv$.

$u = c_1,\ v = c_2$ intersect at right angle $\xrightarrow[\text{under } f]{\text{images}}$ image curves intersect at right angle.

CR - $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ $x = r\cos\theta$, $y = r\sin\theta$.

Polar co-ordinates.

$z = x + iy$
$= re^{i\theta}$.

CR eqn. form.

$$\boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \\ v_r &= -\frac{1}{r} u_\theta \end{aligned}}$$