Lagrange's MVTh.

27.7.16.

$f(x)$ $[a,b]$

→ continuous in $[a,b]$

→ derivable in (a,b).

$\exists$ at least one c, $a<c<b$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Observation 1.

$f'(x) = 0$ when $x \in [a,b]$

$\Rightarrow f(x) = f(a) \ \forall x \in [a,b]$

proof.

$a < x_1 \leq b$.

f' exists $\Rightarrow$ f is cont. in $[a,b]$.

Also f is derivable

$[a,x_1]$ $\cancel{[x_1,b]}$.

$\therefore$ By MVTh.

$\cancel{f'(b)} \cancel{=}$ $\exists$ at least one c,

$c \in (a, x_1)$

s.t. $$\frac{f(x_1) - f(a)}{x_1 - a} = f'(c) = 0$$

$$\Rightarrow \quad f(x_1) = f(a).$$

as $x_1 \in (a, b]$ is any arb. pt.

$$\Rightarrow \quad f(x) = f(a) \quad \forall\, x \in (a, b]$$

$$\forall\, x \in [a, b].$$

<u>Observation 2.</u>

$$\frac{d}{dx}\{F(x)\} = \frac{d}{dx}[G(x)]$$

$\Rightarrow$ $F(x)$, $G(x)$ differ by a const. $\forall x \in [a, b]$.

Use obs. 1.

Observation 3.

Estimating the size of fn! & making numerical approximation.

Example: Use MVTh. to prove that $\sin 46^\circ$ is approximately equal to $\frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$.

Sol^n. Let $f(x) = \sin x$ in $[45^\circ, 46^\circ]$

Continuous derivable

By MVTh. in $[45^\circ, 46^\circ]$,

$$\frac{f(46^\circ) - f(45^\circ)}{46^\circ - 45^\circ} = f'(c), \quad 45^\circ < c < 46^\circ.$$

$$\frac{\sin 46^\circ - \sin 45^\circ}{1^\circ} = \cos c < \cos 45^\circ$$

$1^\circ \rightarrow \frac{\pi}{180}$.

Exercise: $\sqrt[3]{28} = ?$

Using MVTh.

Example: Show that

$$\frac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \frac{v-u}{1+u^2},$$

$0 < u < v.$

and deduce that

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}.$$

Soln. let $\boxed{f(x) = \tan^{-1} x}$, $0 < u < x < v$

$[u, v]$.

apply MVTh.

$$f'(x) = \frac{1}{1+x^2}$$

Both the conditions of Lagrange's MVTh. holds.

$\therefore \frac{f(v) - f(u)}{v-u} = f'(c), \quad c \in (u, v)$

$$\boxed{\frac{\tan^{-1} v - \tan^{-1} u}{v-u}} = \frac{1}{1+c^2}$$

$u < c < v.$

$$\frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$$

replace it. yields the result.

For the deduction.

put $v = \frac{4}{3}$, $u = 1$

Observation 4.

If f is continuous in $[a,b]$ & $f'(x) > 0$ (<0) in (a,b), then f is strictly increasing (dec.) (mon. inc. $\uparrow$) fn^n. in $[a,b]$.

proof. let $a \le x_1 < x_2 \le b$.

$$\boxed{\begin{array}{l}\text{Claim. } f \uparrow \text{ in } [a,b] \\ \text{i.e. } x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \\ \forall x_1, x_2 \in [a,b].\end{array}}$$

Given $f'(x) > 0$. in (a,b).

$\hookrightarrow$ f derivable in (a,b)

Given f continuous in $[a,b]$.

$\therefore$ By Lagrange's MVTh on $f(x)$ over $[x_1, x_2]$

$$\frac{f(x_2) - f(x_1)}{\underbrace{x_2 - x_1}_{>0}} = f'(c), \quad x_1 < c < x_2$$

$$> 0$$

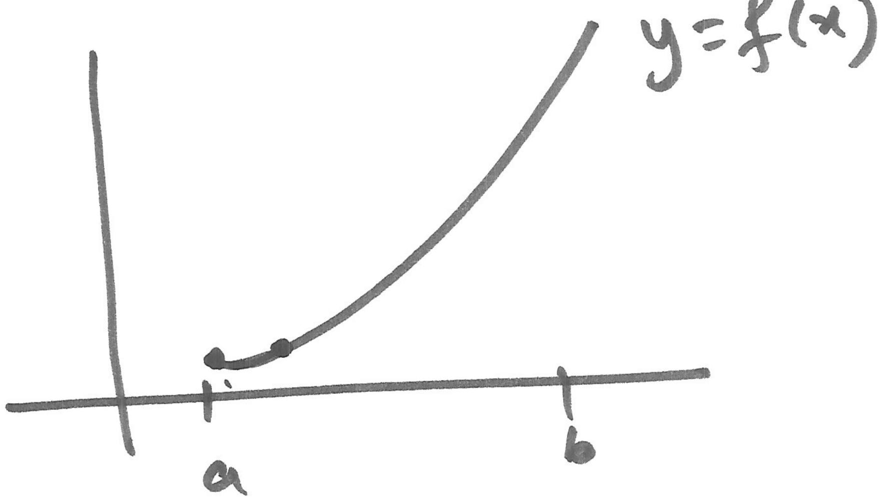
$\Rightarrow f(x_2) > f(x_1)$ as $x_2 > x_1$

Combining.

- If $f'(x) > 0$ in (a, b)

  & $f(a) \geq 0$

  then $\cancel{f'(x)}$ $f(x)$ is +ve throughout (a, b).



- if $f'(x) < 0$ in (a, b) &

  $f(a) \leq 0$.

  then $f(x)$ is −ve fn. in (a, b).

Example: Show that

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$$

$\forall x > 0.$

Sol^n:

Let $\boxed{f(x) = \log(1+x) - x + \frac{x^2}{2}}$.

$$f'(x) = \frac{1}{1+x} - 1 + x.$$

$$= \frac{x^2}{1+x} > 0 \quad \forall x > 0.$$

~~B~~ $(0, \infty)$.

Also $f(0) = 0$.

~~$\therefore f(x)$ is +ve fu^n~~

$f(x) > f(0)$.

$x - \frac{x^2}{2} < \log(1+x)$.

Exercise. Show that $\cos x < \left(\frac{\sin x}{x}\right)^3$, for $0 < x < \frac{\pi}{2}$.

Example: Prove that

$$\frac{\tan x}{x} > \frac{x}{\sin x} \quad \text{for } 0 < x < \frac{\pi}{2}$$

Sol^n. To prove that

$$\frac{\tan x \sin x - x^2}{x \sin x} > 0$$

for $0 < x < \frac{\pi}{2}$

Let $f(x) = \tan x \sin x - x^2$.

$$f'(x) = \tan x \cos x + \sec^2 x \sin x - 2x$$

$$= \sin x + \sec^2 x \sin x - 2x.$$

$$f''(x) = \cos x + \sec^2 x \cos x + 2 \sec x \sec x \tan x \sin x - 2$$

$$= \underline{\cos x + \sec x} + 2 \sin x \tan x \sec^2 x - 2$$

$$= \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2 + \underline{2 \sin x \tan x \sec^2 x}$$

$$> 0 \quad \text{for } 0 < x < \frac{\pi}{2}.$$

$$\Rightarrow f'(x) \text{ is } \uparrow \text{ for } \underline{0 < x < \tfrac{\pi}{2}}.$$

$$\therefore \underline{f'(0) < f'(x)} < f'\left(\tfrac{\pi}{2}\right).$$

$$f'(x) > f'(0) = 0 \quad \text{for } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow f(x) \text{ is } \uparrow \text{ for } 0 < x < \frac{\pi}{2}$$

$$\therefore f(x) > f(0) \quad \text{for } 0 < x < \frac{\pi}{2}.$$

$$\tan x \sin x - x^2 > 0.$$

$\Rightarrow$ ~~tanx si~~ the result.

Example: Show that if $0 < p < q$, then

$$\left(1+\frac{x}{p}\right)^p < \left(1+\frac{x}{q}\right)^q \text{ for } x>0$$

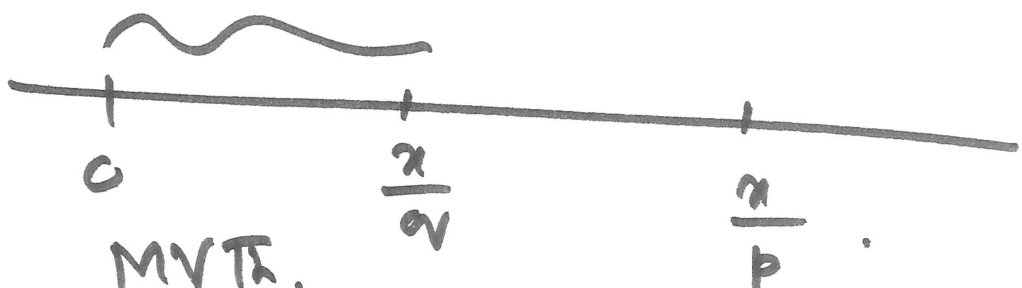
$$\left[\begin{array}{ll} \left(1+\frac{1}{x}\right)^x \uparrow & \forall x>0 \\ \left(1-\frac{1}{x}\right)^x \uparrow & \end{array}\right]$$

Sol^n.

Let $f(x) = \log(1+x)$.

$$f'(x) = \frac{1}{1+x}$$

$0 < p < q$ & $x > 0 \Rightarrow \frac{x}{p} > \frac{x}{q}$



$$\left[0, \frac{x}{q}\right] \xrightarrow[\text{on } f(x)=\log(1+x)]{\text{MVTh.}}$$

(1) — $$\boxed{\frac{\log\left(1+\frac{x}{q}\right) - \log(1)}{\frac{x}{q} - 0} = f'(a_0) = \frac{1}{1+a_0},}$$

for some a_0, $0 < a_0 < \frac{x}{q}$.

$$\left[\frac{x}{q}, \frac{x}{p}\right] \xrightarrow[f(x)=\log(1+x)]{\text{MVTh. on}}$$

(2) — $\dfrac{\log\left(1+\frac{x}{p}\right) - \log\left(1+\frac{x}{q}\right)}{\frac{x}{p} - \frac{x}{q}} = f'(a$ $= \dfrac{1}{1+a_1}$,

for some a_1, $\frac{x}{q} < a_1 < \frac{x}{p}$

$0 < a_0 < \frac{x}{q} < a_1 < \frac{x}{p}$

$a_0 < a_1 \Rightarrow \dfrac{1}{1+a_0} > \dfrac{1}{1+a_1}$

replace bit using (1) — replace using (2).

Simplify $\longrightarrow$ yields the result.

Cauchy's MVTh.

If two fn^{ns}: $f(x)$, $g(x)$

i) are both continuous in $[a, b]$

ii) are both derivable in (a, b)

iii) $g'(x)$ does not vanish at any value of x in (a, b).

then $\exists$ at least one value, say c,

s.t.
$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)},$$

$a < c < b$.