19.10.16

Complex functions.

- $\omega = f(z)$, $\underline{z = x+iy}$

$= u + iv$

when $u = u(x,y)$, $v = v(x,y)$

- **limit** $\boxed{\lim_{z \to z_0} f(z) = l.}$

- **Theorem**
  - $\omega = f(z)$ be defined in a domain D (except perhaps at z_0 in D).
  - $z_0 = x_0 + iy_0$ (x_0, y_0 real).
  - $\omega = f(z)$ is equivalent to real fun. $u(x,y)$ & $v(x,y)$.

Then $\boxed{\lim_{z \to z_0} f(z) = l = a+ib}$ (say)

iff

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = a \quad \& \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = b.$$

proof.

$$\lim_{z\to z_0} f(z) = \underset{a+ib}{\underset{\parallel}{l}} \Rightarrow \lim_{z\to z_0} \overline{f(z)} = \underset{a-ib}{\underset{\parallel}{\bar{l}}}$$

$$\Rightarrow \lim_{z\to z_0} \mathrm{Re}\, f(z) = \lim_{z\to z_0} \left(\frac{f(z)+\overline{f(z)}}{2} \right)$$

$$= \frac{l+\bar{l}}{2} = a$$

$$\lim_{z\to z_0} \mathrm{Im}\, f(z) = b.$$

as $\mathrm{Im}\, f(z) = \dfrac{f(z)-f(\bar{z})}{2i}$

$$\frac{\cancel{u}+iv-(\cancel{u}-iv)}{2i}$$

$= v$.

- **Continuity of $w=f(z)$ at $z=z_0$**

i) $\lim_{z\to z_0} f(z) = l$ must exist

ii) $f(z)$ must be defined at $z=z_0$

iii) $l = f(z_0)$.

~~Exam~~

- Continuity in a domain D
- if $f(z)$ is continuous at every pt. in D

Theorem.

i) Sum, product, quotient (except divisibi- by 0). of continuous fn.
$\xrightarrow{\text{yields}}$ continuous fn's.

ii) Continuous fn. of a continuous fn. is again continuous fn.

iii) if $f(z)$ continuous in a closed region, then it is bdd. in that region.

$$|f(z)| < M \quad \forall z \in D$$

region

M real +ve const.

iv) $f(z)$ continuous $\Rightarrow$ its real & imaginary parts are continuous.

Examples.

a) all poly^{s}.

b) e^{z}

c) $\sin z$, $\cos z$.

Example: Is the f^n.

$$f(z) = \begin{cases} z^2, & z \neq z_0 \quad (z_0 \neq 0) \\ 0, & z = z_0 \end{cases}$$

continuous at $z = z_0$?

Ans. No, has removable discontinuity at $z = z_0$.

Example: $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$

continuous at $z = i$?

Soln. $\lim\limits_{z \to i} f(z) = 4 + 4i$

$$z - i \,\Big|\, 3z^4 - 2z^3 + 8z^2 - 2z + 5$$

$\vdots$

$$0$$

Example: Discuss continuity of the following fn

i) $f(z) = \dfrac{z}{z^2+1}$

ii) $f(z) = \operatorname{cosec} z.$

Example:

a) $$f(z) = \begin{cases} \dfrac{xy^3}{x^2+y^6}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

b) $$f(z) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}}, & z \neq 0 \\ 0, & z = 0. \end{cases}$$

$z = x + iy$

Complex Differentiation: Analytic fun^{ns}.

- $f(z) \to$ single-valued fun^n. defined on a domain D of the complex plane $\mathbb{C}$.

- f differentiable at $z_0 \in D$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z)$$

exists, is finite & is independent of the manner in which $z \to z_0$.

or

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z)$$

or

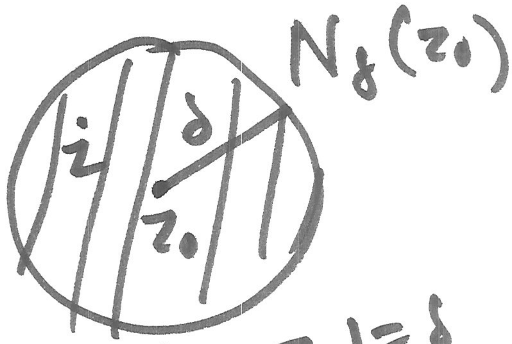
$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z).$$

Note: h or Δz complex nos.

$z = x + iy$, $\Delta z = \Delta x + i\,\Delta y$.

Analytic fn.

A fn. f is said to be analytic (or holomorphic or regular) at a pt. z_0, if $\exists$ some δ-nbd. of z_0 at all pts. of which $f'(z)$ exist.



$N_\delta(z_0)$: $|z - z_0| = \delta$

- Singular pts. $\rightarrow$ when the fn. fails to be analytic.

Continuity & Differentiability.

differentiability at z_0 of $f(z)$ $\Rightarrow$ Continuity at z_0 of $f(z)$.

$f'(z_0)$ exists.

$\cancel{f(z+a)}$

$$f(z_0+h) - f(z_0) = \frac{f(z_0+h) - f(z_0)}{h} \cdot h$$

$$\therefore \lim_{h \to 0} \left\{ f(z_0+h) - f(z_0) \right\}$$

$$= \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \cdot \lim_{h \to 0} h$$

$$= f'(z_0) \cdot 0 = 0.$$

$$\Rightarrow \lim_{h \to 0} f(z_0+h) = f(z_0)$$

$\Rightarrow f$ Continuous at z_0.

• Continuity does not always imply differentiability.

<u>Example:</u> $f(z) = \bar{z}$

continuous, nowhere analytic.

<u>Sol^n:</u>

$$\frac{d}{dz}(\bar{z}) = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h}$$

$z = x + iy$

$h = \Delta x + i \Delta y$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{(x + \Delta x + i(y + \Delta y))} - \overline{(x+iy)}}{\Delta x + i \Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\cancel{x} + \Delta x - i(\cancel{y} + \Delta y) - (\cancel{x} - i\cancel{y})}{\Delta x + i \Delta y}.$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overbrace{\Delta x - i \Delta y}^{\bar{h}}}{\underbrace{\Delta x + i \Delta y}_{h}} = \begin{cases} 1 & \text{along } x\text{-axis} \\ -1 & \text{along } y\text{-axis} \end{cases}$$

Example: $f(z) = |z^2|$.

continuous $\forall z \in \mathbb{C}$,

but is differentiable nowhere except at the origin.

$z = x + iy$

$|z|^2 = x^2 + y^2$

Sol^n: When $z \neq z_0$, $z_0 \neq 0$, we have

$$\frac{|z|^2 - |z_0|^2}{z - z_0} = \frac{z\bar{z} - z_0\bar{z_0}}{z - z_0}$$

$$= \frac{\bar{z}(z - z_0) + z_0(\bar{z} - \bar{z_0})}{z - z_0}.$$

$$= \bar{z} + z_0 \frac{\bar{z} - \bar{z_0}}{z - z_0}.$$

let $z - z_0 = re^{i\theta}$

Then $\dfrac{|z|^2-|z_0|^2}{z-z_0} = \bar{z} + z_0 \dfrac{r e^{-i\theta}}{r e^{i\theta}}$.

$$= \bar{z} + z_0 e^{-2i\theta}.$$

$$= \bar{z} + z_0 (\cos 2\theta - i \sin 2\theta).$$

$\not\Rightarrow$ a unique limit as $z \to z_0$ in any manner.

But when $z_0 = 0$, then differentiable.

$\Rightarrow$ $|z|^2$ is nowhere analytic except at the origin.

Some common examples of complex differentiability

1. $f(z) = z^n$, n is +ve integer
   - $\to$ diff. $\forall$ pts. in $\mathbb{C}$.
   - $\to$ analytic over the entire complex plane.

2. $f(z) = \text{Re } z$ and $f(z) = \text{Im } z$.

$\rightarrow$ not differentiable.
(Exercise).

3. $f(z) = \frac{1}{z}$ $\rightarrow$ diff. $\forall z \in \mathbb{C}$
except at $(0,0)$.

4. $f(z) = \bar{z}$ $\rightarrow$ diff. nowhere in $\mathbb{C}$
however continuous
everywhere in $\mathbb{C}$.

do it using polar
co-ordinates.

5. $f(z) = |z^2|$ $\rightarrow$ diff. only at
$z = 0$.
$\rightarrow$ however continuous
at every pt. z.

Necessary condition for a fu^n: $f(z)$ to be analytic

(Cauchy- Riemann equ^n).

$f(z) = u + iv$, $z = x + iy$.

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} \text{CR equ}^{ns}.$$

$u_x, u_y, v_x, v_y \rightarrow$ 1st. order partial derivative of u, v.

- $z = x + iy$
- $f(z) = u(x,y) + i\, v(x,y)$, u, v are real fu^{ns}.
- if $f(z)$ is diff. at z, then

$$\frac{f(z+h) - f(z)}{h} \rightarrow \text{a unique limit}$$

as $h \rightarrow 0$

· if we take h to be real, then

$z = x + iy$

$z + h = (x+h) + iy$

$f(z) = u(x,y) + iv(x,y)$

$f(z+h) = u(x+h, y) + iv(x+h, y)$

$$\frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h}$$

$$\boxed{f'(z) = u_x + iv_x.} \quad \text{--- (1)}$$

· if we take h to be imaginary,

$$\boxed{f'(z) = -iu_y + v_y.} \quad \text{--- (2)}.$$

(1), (2) $\Rightarrow$ $\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$ CR eq^{ns}