Complex Analysis.

- complex numbers $\longrightarrow$ $z = x + iy$, $x, y \in \mathbb{R}$.

1-1 correspondence.

$$\mathbb{C} \longleftrightarrow \mathbb{R}^2$$

$$x + iy \longleftrightarrow (x, y).$$

$\mathbb{C} \longrightarrow$ complex plane (z-plane)
/ Argand plane
/ Gaussian plane.

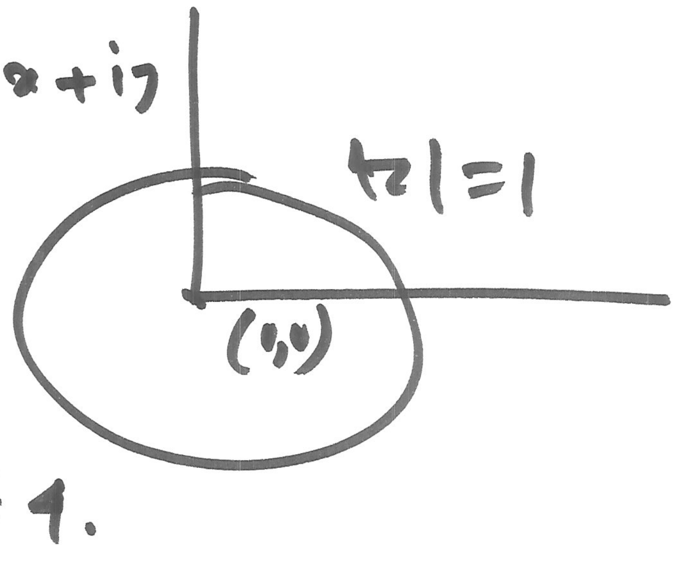
Examples.

1. $\boxed{|z| = 1.}$

$$\Updownarrow$$

$$x^2 + y^2 = 1$$

$z = x + iy$



2. $|z-1| + |z+1| \leq 4.$

$$\downarrow \qquad z = x + iy$$

$$\frac{x^2}{4} + \frac{y^2}{3} \leq 1. \quad \text{(check).}$$

3. $\arg \frac{z-1}{z+1} = \pi/3 \rightarrow$ in xy-plane ??

4. $|z^2 - 1| < 1$.

5. $\boxed{\left|\frac{z-i}{z+i}\right| \geq 2.}$ $\rightarrow$ interior ~~o~~f boundary of the circle

$3x^2 + 3y^2 + 10y + 3 = 0.$

Definitions.

- Neighbourhood of a point z_0

  - δ-nbd. of z_0 $\rightarrow$ $N_\delta(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \delta\}$

  - deleted δ-nbd. of z_0

    $N'_\delta(z_0) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \delta\}$.

- Limit pts ~~of~~ of a set S in $\mathbb{C}$.

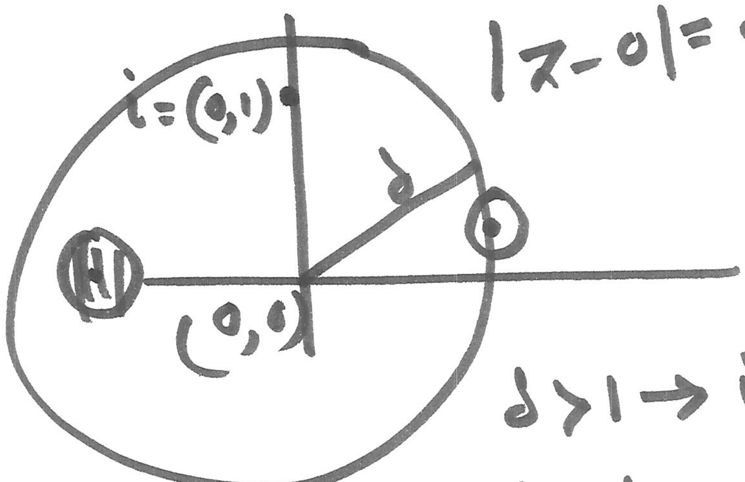
  / point of accumulation.

z_0 is a limit pt. of S if every <u>deleted</u> δ-nbd. of z_0 contains at least one point of S, where δ is arbitrarily small +ve no.

<u>Example:</u> $S = \left\{ \frac{i}{n} \;\middle|\; n = 1, 2, 3, \ldots \right\}$.

$z_0 = 0$ is a limit pt. of S.

$|z - 0| = \delta$.



$\delta > 1 \rightarrow i \in S$.

$\delta > \frac{1}{2} \rightarrow \frac{i}{2} \in S$.

Note that $z_0 = 0$ is not in S.

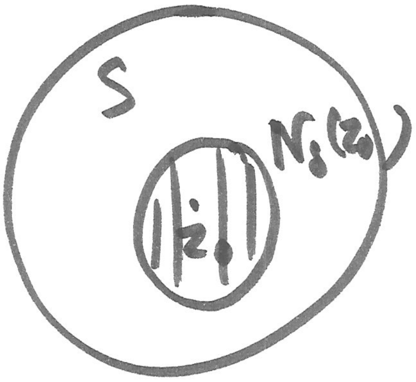
<u>Closed set</u>. S is said to be closed if it contains all its limit pts.

$S = \left\{ \frac{i}{n} \;\middle|\; n = 1, 2, \ldots \right\}$ not closed.

neither an open set

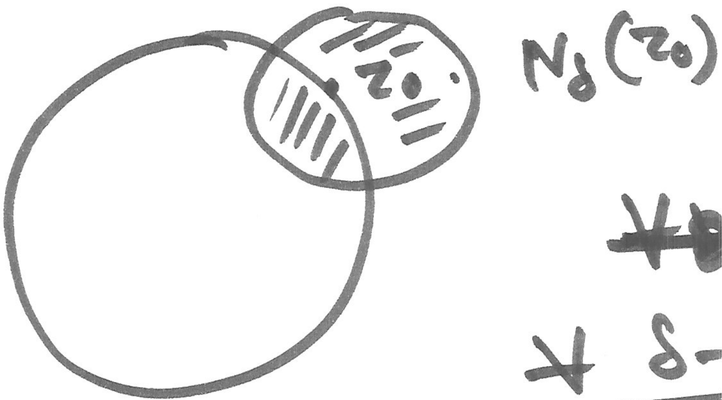
Interior, Exterior & Boundary pts.

- $z_0 \to$ interior pt. of S.



$\exists$ some δ-nbd. $N_\delta(z_0)$ of z_0 s.t. all pts. in $N_\delta(z_0)$ $\in S$.

- $z_0 \to$ boundary pt. of S.



$\forall$ δ-nbd. $N_\delta(z_0)$ of z_0, $N_\delta(z_0)$ contains pts. of S as well as pts. not of S.

- Otherwise, z_0 is an exterior pt.

Open set $\to$ contains only interior pts. of the set.

Example:

i) $\{ z \in \mathbb{C} \mid |z| < 1 \}$.

open set.

ii) $S = \{ \frac{i}{n} \mid n = 1, 2, \cdots \}$

not an open set.

Closure of a set.

$\overline{S} = S \cup \{ \text{all the limit pts. of } S \}$.

$\overline{S} = S \cup \{0\}$.

Compact set $\longrightarrow$ bdd. set. which is closed.

(closed bdd. set).

$S = \{ \frac{i}{n} \mid \underline{n = 1, 2, \cdots} \}$ $\underline{\text{bdd. set.}}$

$\forall z \in S, \ |z| < M,$

M is a real const.

$\left| \frac{i}{n} \right| < \frac{1}{n} \leq 1.$

$n \geq 1$

functions of complex variables.

- $\omega = f(z)$, $z \in D$, $D \to$ a domain in the complex plane.

e.g. $\boxed{f(z) = z^2.}$

$f(3i) = (3i)^2 = -9.$

to each value $z \in D$, there corresponds one or more value of ω by the given rule(s) f

$\omega = f(z)$ $\to$ single valued
$\to$ multiple-valued fn's.

e.g. $\omega = z^{1/3} \to 3$ branches

$\omega = \ln z \to$ infinitely many branches

a collection of single-valued fn's each of which is called a branch.

- A particular branch is taken to be the ~~principle~~ principal branch.

$$\omega = z^{1/3} \longrightarrow \underline{0 \le \theta < 2\pi}$$

$$z = re^{i\theta}$$

$$= r(\cos\theta + i\sin\theta).$$

$$= re^{i(\theta + 2k\pi)}, \quad k = 0, 1, 2.$$

Mapping or Transformation.

~~$\omega = x+iy$~~

- $w = u + iv$. (u, v are real).

$\rightarrow$ single valued fu^n. of

$z = x+iy$.

Then $w = f(z)$ can be written as

$u+iv = f(x+iy)$.

$\Downarrow$

equating real & imaginary part.

$u = u(x,y), \quad v = v(x,y).$

e.g. $\omega = z^2 = u + iv$, $z = x + iy$

$= (x+iy)^2$

$= \underbrace{x^2 - y^2} + \underline{2ixy}$.

$u(x,y) = u = x^2 - y^2$, $v = 2xy$.

$\qquad\qquad\qquad\qquad\quad \parallel$

$\qquad\qquad\qquad\quad v(x,y)$.

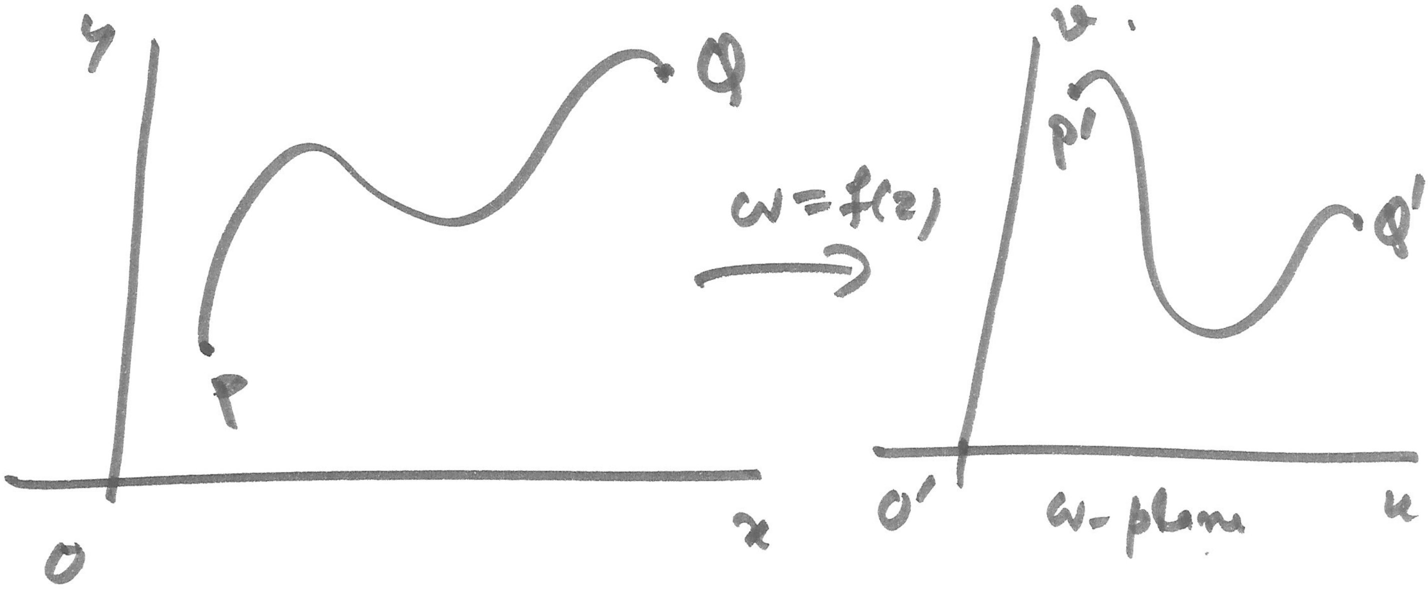
i.e. complex fn. $\omega = f(z)$ is equivalent to two real fn. $u = u(x,y)$, $v = v(x,y)$

$P(1,2)$ in z-plane

$z = 1 + 2i \xrightarrow{\omega = z^2} \omega = u + iv$.

$= -3 + i4$

$P(1,2)$ in z-plane $\xrightarrow{\omega = z^2}$ $P'(-3,4)$. in ω-plane.



z-plane.

$z = x + iy$

$w = u + iv$.

$w = f(z)$ or $\left\{ \begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array} \right.$

is called a transformation / mapping.

· Curvilinear co-ordinates.

$$\begin{aligned} w &= f(z) \longrightarrow \left[\begin{array}{l} u = u(x,y) \\ v = v(x,y) \end{array} \right. \\ &= u + iv \end{aligned}$$

$z = x + iy$

$(x,y) \to$ rectangular co-ordinate, corr. to $P(x,y)$

$(u,v) \to$ curvilinear co-ordinate corr. to P.

Example: $\omega = f(z) = z^2 = u + iv$

or

equivalently, $u = u(x,y) = x^2 - y^2$

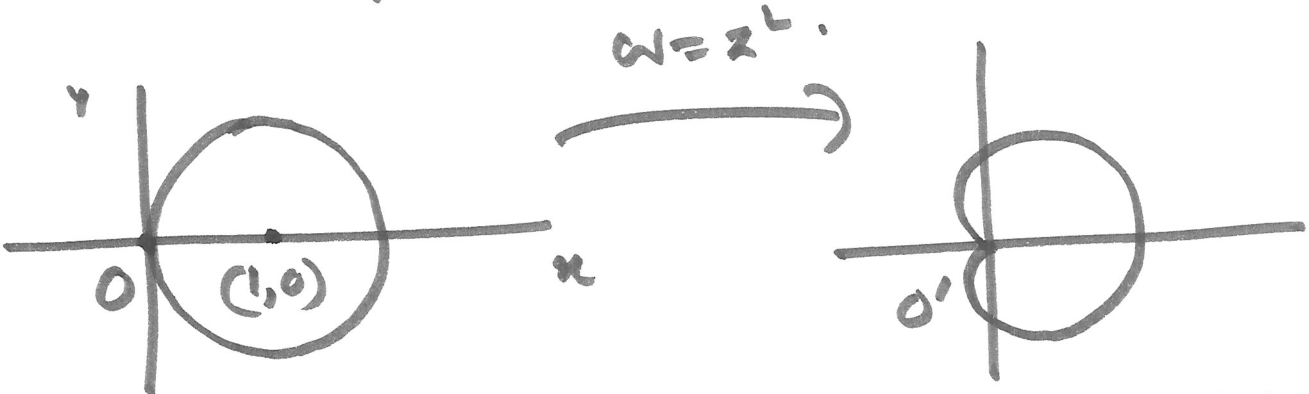
$v = v(x,y) = 2xy$.

$$\omega = z^2.$$

$$D:\ |z-1| < 1 \longrightarrow D' \ ?$$

domain in z-plane

domain in ω-plane.



circle in z-plane

cardioid in ω-plane.

Elementary fn's.

1. Polynomial fn.

$$\omega = f(z) = \underline{\underline{a_0}}\, z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n.$$

$a_0 \neq 0$, $a_1, \ldots, a_n$ all complex no's.

particular case $n=1$

linear fn.

$\boxed{w = az+b}$, a, b complex constants.

2. Rational algebraic fns.

$$w = \frac{f(z)}{g(z)}, \quad f(z), g(z) \text{ poly}^{s}.$$

special case.

$$w = \frac{az+b}{cz+d}, \quad ad-bc \neq 0.$$

↳ bilinear transformation

3. $\boxed{w = e^z}$ $\sin z$, $\cos z$, $\sec z$, ...

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

4. <u>Logarithmic fn.</u> → inverse of exp. fn.

$\omega = \ln z = \ln r + i(\theta + 2k\pi)$,

$k = 0, \pm 1, \pm 2, \dots$

$z = re^{i\theta}$

→ multiple valued fn. (infinitely many branches).

$\underline{0 \leq \theta < 2\pi}$ → principal branch.

(any other interval of length 2π can be taken as principal branch. e.g. $-\pi \leq \theta < \pi$).

5. $\sin^{-1} z$, $\cos^{-1} z$ → multiple valued fn.

~~Sin~~ =

$\sin^{-1} z = \frac{1}{i} \ln\left(iz + \sqrt{1-z^2}\right) + 2k\pi i$,

$k = 0, \pm 1, \pm 2, \dots$

· All the ~~fn~~. aforementioned fn^{ns}.

$$\downarrow \; +, -, \times, \div$$

derived fn^{ns}.

all are elementary fn^{ns}.

Algebraic fn^{ns}.

· if ω is a sol^n. of the poly^s. eq^n.

① — $\underline{P_0(z)}\,\omega^n + P_1(z)\omega^{n-1} + \cdots + P_{n-1}(z)\omega + P_n(z) = 0.$

when $P_0(z) \neq 0$, $P_1(z), \ldots, P_n(z)$ are poly^s in z & n is a +ve integer.

e.g. $\omega = z^{1/2}$ is a sol^n. of $\underline{\omega^2 - z = 0.}$

$\omega^2 = z$.

Any fn^{ns}. which cannot be expressed as a sol^n. of ①, is called trancendental fn^{ns}.

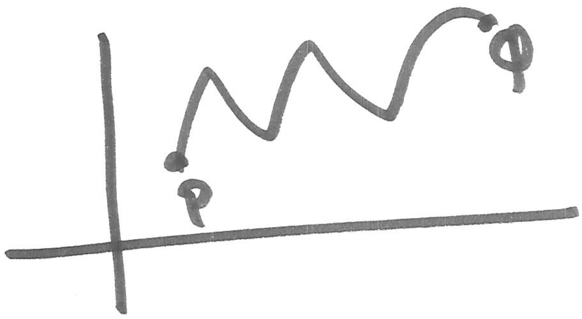
$\omega = \arg z = \tan^{-1}\frac{y}{x}$, $\quad z = re^{i\theta}$, $\theta = \tan^{-1}\frac{y}{x}$.

$-\pi < \theta \leq \pi$ $\quad = re^{i(\theta + 2k\pi)}$

Limit & Continuity.

· Limit

. $\omega = f(z)$ is defined in a domain D.
(except perhaps at $z_0 \in D$).



· l be a complex const.

open domain
$\longrightarrow$ open connected set.

$\downarrow$

S

closed domain
$\longrightarrow$

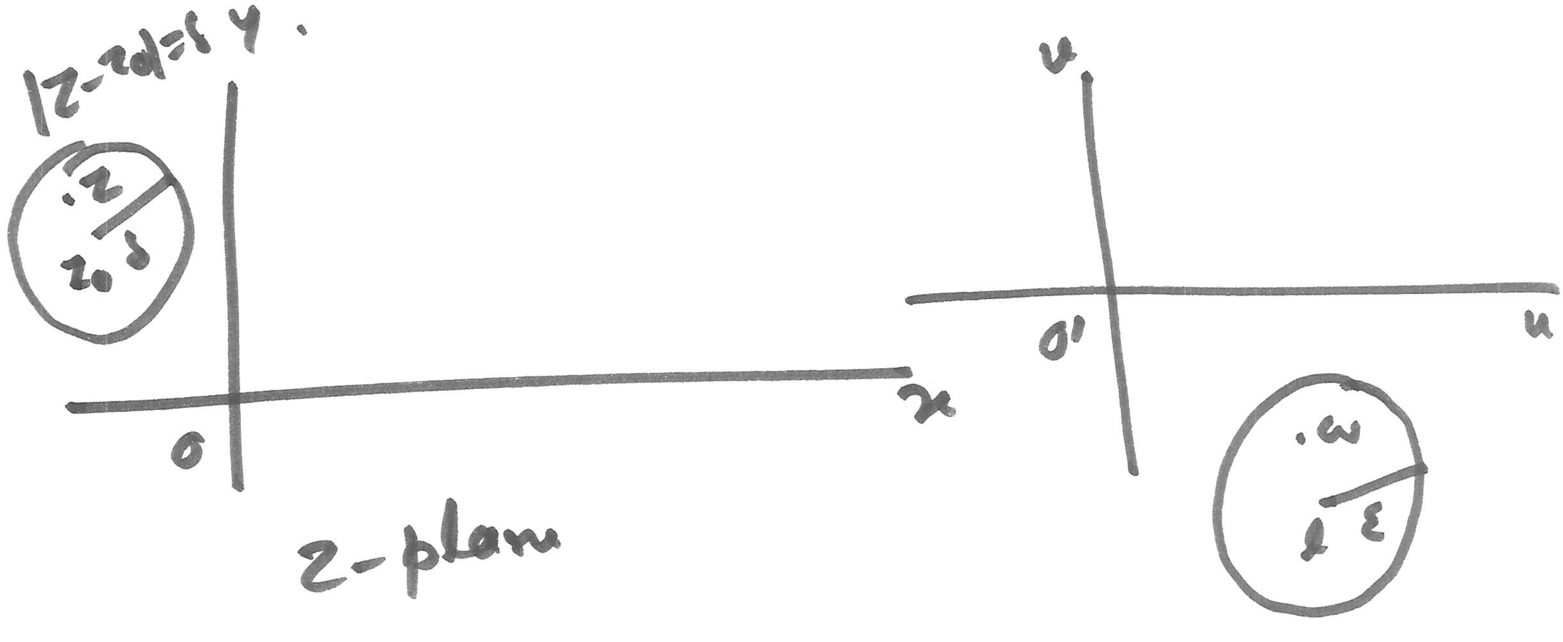
· $\lim_{z \to z_0} f(z) = l$ if

to each +ve ε (arb. small), $\exists$ a +ve. δ s.t.

$|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

- l is the limit of $f(z)$ as $z \to z_0$ along any path whatsoever.

i.e., $\exists$ a deleted δ-nbd. of z_0 in which $|f(z)-l|$ can be made as small as we like.



z-plane

- the limit l should be unique & independent of path along which $z \to z_0$.

Example: $\lim_{z \to 0} \frac{\bar{z}}{z}$ $z = x+iy$.

$z \to 0$ along x-axis $y=0$

$$\lim_{z \to 0} \frac{x-iy}{x+iy} = \lim_{z \to 0} \frac{x}{x} = 1.$$

$z \to 0$ along y-axis $x=0$.

$$\lim_{z \to 0} \frac{-iy}{iy} = -1.$$

So limit does not exist.

Example: Prove that $\lim_{z \to z_0} f(z) = z_0^2$

if $f(z) = z^2$.

Sol^n: To prove that $\forall \varepsilon > 0$, $\exists \delta > 0$ (δ generally depends on ε) s.t.

$|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Let $\boxed{\delta \leq 1.}$

Now whenever $0 < |z - z_0| < \delta$, we get

$$
\begin{aligned}
|z^2 - z_0^2| &= |z - z_0|\,|z + z_0| \\
&< \delta\,|z + z_0| \\
&= \delta\,|z - z_0 + 2z_0| \\
&\leq \delta\{|z - z_0| + 2|z_0|\} \\
&\leq \delta\{1 + 2|z_0|\} < \varepsilon
\end{aligned}
$$

whenever $\boxed{\delta < \dfrac{\varepsilon}{1 + 2|z_0|}}$.

Take $\delta = \min\left\{1, \dfrac{\varepsilon}{1 + 2|z_0|}\right\}$.

Then $|z^2 - z_0^2| < \varepsilon$ whenever $0 < |z - z_0| < \delta$

& the result follows.

<u>Example:</u> If $\lim_{z\to z_0} f(z)$ exists, then it must be unique.

<u>Sol^n.</u> If not, let $\lim_{z\to z_0} f(z) = \begin{cases} l_1 \\ l_2 \end{cases}$

<u>Claim</u> $l_1 = l_2$

Let $\varepsilon > 0$ be given.

Then a suitably chosen δ can be found s.t

$$|f(z) - l_1| < \frac{\varepsilon}{2} \quad \forall z \text{ in } 0 < |z-z_0| < \delta$$

& $$|f(z) - l_2| < \frac{\varepsilon}{2} \quad \forall z \text{ in } 0 < |z-z_0| < \delta.$$

$$\therefore |l_1 - l_2| = |l_1 - f(z) + f(z) - l_2|$$

$$\leq |l_1 - f(z)| + |f(z) - l_2|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{when } \varepsilon > 0.$$

no matter how small.

$\Rightarrow |l_1 - l_2| = 0$. i.e. $l_1 = l_2$.

Example: If $\boxed{\lim_{z \to z_0} g(z) = m \; (\neq 0)}$,

then prove that $\exists\, \delta > 0$ s.t.

$$|g(z)| > \frac{1}{2}|m| \text{ for } 0 < |z - z_0| < \delta.$$

Sol^n: $\lim_{z \to z_0} g(z) = m$

$\Rightarrow \exists\, \delta > 0$ s.t. $|g(z) - m| < \frac{1}{2}|m| = \varepsilon$.

whenever z is in $0 < |z - z_0| < \delta$

Now $|m| = |m - g(z) + g(z)|$

$$\leq |m - g(z)| + |g(z)|$$

$$< \frac{1}{2}|m| + |g(z)|.$$

$\Rightarrow |g(z)| > |m| - \frac{1}{2}|m| = \frac{1}{2}|m|$.

for $0 < |z - z_0| < \delta$.

Theorems on limits.

Given $\lim_{z\to z_0} f(z) = l$, $\lim_{z\to z_1} g(z) = m$.

Then prove that

a) $\lim_{z\to z_0} [f(z) \pm g(z)] = l \pm m$

→ b) $\lim_{z\to z_0} [f(z)\, g(z)] = lm$

→ c) $\lim_{z\to z_0} \left[\frac{1}{g(z)}\right] = \frac{1}{m}$ if $m \neq 0$.

d) $\lim_{z\to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{l}{m}$ $(m\neq 0)$.

Example: $\lim_{z\to -2i} \frac{(2z+3)(z-1)}{z^2-2z+4} = -\frac{1}{2}+\frac{11}{4}i$ (check).

Example: $\lim_{z\to 2e^{\pi i/3}} \frac{(z^3+8)(z^2-4)}{(z^4+4z^2+16)(z^2-4)}$

$= \lim_{z\to 2e^{\pi i/3}} \frac{(z^3+8)(z^2-4)}{(z^6-64)}$

$$= \lim_{z \to 2e^{\pi i/3}} \frac{z^2-4}{z^2-8}$$

$$= \frac{1}{8}\left(3 - i\sqrt{3}\right).$$

$$\frac{4e^{\frac{2\pi i}{3}}-4}{4e^{2\pi i/3}-8}.$$

- $\boxed{\lim_{z \to \infty} f(z) = l}$?

if for any $\varepsilon > 0$ (arb. small), we can find a real const. $M > 0$ s.t.

$|f(z) - l| < \varepsilon$ whenever $|z| > M$.

- $\boxed{\lim_{z \to z_0} f(z) = \infty}$?

if for any $N > 0$ (arb. large), we can find $\delta > 0$ s.t.

$|f(z)| > N$ whenever $0 < |z - z_0| < \delta$.

<u>Infinity.</u>

- pt. $z = 0 \xrightarrow[\text{transformatn}]{w = \frac{1}{z}} w = \infty$.

  pt. at infinity in w-plane.

- pt. $z = \infty \xrightarrow{w = \frac{1}{z}} w = 0$.

Behaviour of $f(z)$ at $z = \infty$.

$\longrightarrow$ examine the behaviour of $f\left(\frac{1}{w}\right)$ at $w = 0$.