System of Simultaneous linear equations.

5.10.16

Example:

$$\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0 \\ \dfrac{dy}{dt} + 5x + 3y = 0. \end{cases}$$

Sol^n:

$$D = \frac{d}{dt}.$$

$$\begin{cases} (D+2)x + (D+1)y = 0. \\ 5x + (D+3)y = 0. \end{cases}$$

$$5(D+2)x + 5(D+1)y = 0.$$

$$5(D+2)x \pm (D+2)(D+3)y = 0.$$

$$-\qquad\qquad -$$

$$(D^2+1)y = 0. \qquad m = \pm i$$

$$\boxed{y = A\cos t + B\sin t,} \quad A, B \text{ arb. consts.}$$

$$x = -\frac{1}{5}(D+3)y$$

$$= -\frac{1}{5}\left[\quad \cdots \quad\right]$$

Example:

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$$

$$\frac{dy}{dx} = \frac{2xy}{x^2-y^2-z^2}$$

$$\frac{dz}{dx} = \frac{2xz}{x^2-y^2-z^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2-y^2-z^2) + y(2xy) + z(2xz)}.$$

$$= \frac{\frac{1}{2} d(x^2+y^2+z^2)}{x[x^2-y^2-z^2+2y^2+2z^2]}$$

$$= \frac{d(x^2+y^2+z^2)}{2x(x^2+y^2+z^2)}$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c. \quad (1)$$

$$\boxed{y = cz,} \quad c \text{ const}.$$

$$\frac{dz}{\cancel{2x}z} = \frac{d(x^2+y^2+z^2)}{\cancel{2x}(x^2+y^2+z^2)}.$$

$$\log z = \log(x^2+y^2+z^2) + \log c', \quad c' \text{ const}$$

$$\Rightarrow \boxed{x^2+y^2+z^2 = \frac{z}{c'}} \quad -(2).$$

①, ② gives the complete solⁿ.

__Exercise.__

$$\left.\begin{aligned} \frac{d^2x}{dt^2} - 3x - 4y = 0 \\ \frac{d^2y}{dt^2} + x + y = 0 \end{aligned}\right\}$$

or

$$\left.\begin{aligned} (D^2-3)x - 4y = 0. \\ x + (D^2+1)y = 0. \end{aligned}\right\} \quad D \equiv \frac{d}{dt}.$$

__Exercise__

$$\frac{dx}{mz-ny} = \frac{dy}{nz-lz} = \frac{dz}{ly-mx}.$$

__Example:__

$$\text{①}\left\{\begin{aligned} \frac{dx}{dt} &= y+z \\ \frac{dy}{dt} &= x+z \\ \frac{dz}{dt} &= x+y. \end{aligned}\right.$$

Try to reduce the system of eqⁿˢ. ① to a single eqⁿ. with const. coefficients. in single variable

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} + \frac{dz}{dt}.$$

$$= x + z + x + y.$$

$$= 2x + z + y$$

$$= 2x + \frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0.$$

$$(D^2 - D - 2)x = 0. \quad \cancel{y}$$

$$x = e^{mt}.$$

$$m^2 - m - 2 = 0.$$

$$m = -1, 2.$$

$$\boxed{x = c_1 e^{-t} + c_2 e^{2t},} \quad c_1, c_2 \text{ arb. consts.}$$

$$\frac{dx}{dt} = -c_1 e^{-t} + 2c_2 e^{2t}.$$

$$y + z = -c_1 e^{-t} + 2c_2 e^{2t}.$$

$$\boxed{y = -(c_1 + c_3) e^{-t} + c_2 e^{2t}.}$$

$$\boxed{z = c_3 e^{-t} + c_2 e^{2t}}$$

Exercise. Find the general sol^n of the following system of DEs.

$$\left.\begin{aligned} \frac{d^2y}{dx^2} &= z \\ \frac{d^2z}{dx^2} &= y. \end{aligned}\right\} .$$

highest derivative having deg. 1

System of homogeneous linear DE with const. co-efficients.

$$(1)\quad \begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ \dfrac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n. \end{cases}$$

- n equns. in n unknowns $x_1, \ldots, x_n$
- a_{ij} are consts.

Approach I. Reducing (1) to a single equn. of order n in single variable which is linear. hom. equn. with const. coefficients.

Solve it →

Approach II. (without reducing (1) to an equn. of n-th. order).

- Seek for a particular soln. of (1) in the following form.

$$\boxed{x_1 = \alpha_1 e^{kt}, \quad x_2 = \alpha_2 e^{kt}, \ldots, \quad x_n = \alpha_n e^{kt}} \quad (2).$$

$$\begin{cases} k\alpha_1 e^{kt} = (a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n)e^{kt} \\ k\alpha_2 e^{kt} = (a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)e^{kt} \\ \vdots \\ k\alpha_n e^{kt} = (a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n)e^{kt}. \end{cases}$$

$$③ - \begin{cases} (-k+a_{11})\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n = 0 \\ a_{21}\alpha_1 + (-k+a_{22})\alpha_2 + \cdots + a_{2n}\alpha_n = 0 \\ \vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + (a_{nn}-k)\alpha_n = 0. \end{cases}$$

$$\begin{pmatrix} -k+a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & -k+a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & & -k+a_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- $$\Delta(k) = \begin{vmatrix} -k+a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & -k+a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & & -k+a_{nn} \end{vmatrix}$$

- $\Delta(k) \neq 0 \Rightarrow$ (3) has only trivial sol^n. $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

  $x_1(t) = 0 = x_2(t) = \dots = x_n(t)$

  only trivial sol^n.

- (3) has non-trivial sol^n only for k s.t. $\boxed{\Delta(k) \cancel{\neq} = 0.}$

  (4).

  auxiliary eqn.

  real, distinct ✓ — Complex, distinct ✓ — equal. (Real/complex)

A) The roots of the auxiliary eqn. are real & distinct.

- $\Delta(k) = 0 \rightarrow k_1, k_2, \ldots, k_n$ (all distinct, real).

$k_i \rightarrow \alpha_1^{(i)}, \alpha_2^{(i)}, \ldots, \alpha_n^{(i)}$

$$\begin{pmatrix} \Delta(k) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$$\rightarrow \boxed{x_1^{(i)} = \alpha_1^{(i)} e^{k_i t}, \; x_2^{(i)} = \alpha_2^{(i)} e^{k_i t}, \ldots, x_n^{(i)} = \alpha_n^{(i)} e^{k_i t}}$$

General soln. of (1).

$$\left.\begin{aligned} x_1 &= \sum_{i=1}^{n} c_i \alpha_1^{(i)} e^{k_i t} \\ x_2 &= \sum_{i=1}^{n} c_i \alpha_2^{(i)} e^{k_i t} \\ &\vdots \\ x_n &= \sum_{i=1}^{n} c_i \alpha_n^{(i)} e^{k_i t} \end{aligned}\right\} \; c_1, c_2, \ldots, c_n \text{ arb. consts.}$$

Example:

$$\left.\begin{aligned}\frac{dx_1}{dt} &= 2x_1 + 2x_2 \\ \frac{dx_2}{dt} &= x_1 + 3x_2\end{aligned}\right\} \quad \text{--- (1).}$$

Solve using eigen-value approach.
without reducing (1) to an eqn. of higher order in a single variable.

$\Delta(k) = 0.$

$\downarrow$ find roots

$\downarrow$

find $\alpha_1, \ldots, \alpha_n$ cor. to each k.

$$\cancel{\Delta(k) = \left| -k+2 \right.}$$

$$\Delta(k) = \begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0.$$

$k = 1, 4.$

$\underline{k_1 = 4}$

$$\begin{pmatrix} 2-4 & 2 \\ 1 & 3-4 \end{pmatrix} \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$-2\alpha_1^{(1)} + 2\alpha_2^{(1)} = 0.$$

$$\alpha_1^{(1)} - \alpha_2^{(1)} = 0 \Rightarrow \alpha_1^{(1)} = \alpha_2^{(1)}.$$

Take $\alpha_1^{(1)} = 1.$ can be set any arb. const.

$$\Rightarrow \alpha_2^{(1)} = 1.$$

$$\therefore \boxed{x_1^{(1)} = \alpha_1^{(1)} e^{k_1 t} = e^{4t}}, \quad x_2^{(1)} = \alpha_2^{(1)} e^{k_1 t} = e^{4t}.$$

$k_2 = 1$

$$\begin{pmatrix} 2-1 & 2 \\ 1 & 3-1 \end{pmatrix} \begin{pmatrix} \alpha_1^{(2)} \\ \alpha_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$\boxed{x_1^{(2)} = e^t}$, $\quad x_2^{(2)} = -\frac{1}{2} e^t.$

$\therefore$ General soln. of (1) in

$$x_1 = c_1 e^{4t} + c_2 e^t.$$

$$x_2 = c_1 e^{4t} - \frac{c_2}{2} e^t.$$

Example:

$$\frac{d^2x}{dt^2} = a_{11} x + a_{12} y$$

$$\frac{d^2y}{dt^2} = a_{21} x + a_{22} y.$$

auxiliary eqn.

$$\begin{vmatrix} a_{11}-k^2 & a_{12} \\ a_{21} & a_{22}-k^2 \end{vmatrix} = 0.$$

using soln. of the type.

$$x = \alpha e^{kt}, \quad y = \beta e^{kt}.$$

<u>Case B.</u> Complex, ~~unequ~~ unequal roots.

$$k_1 = \alpha + i\beta, \quad k_2 = \alpha - i\beta.$$

$\downarrow$ cor. soln.

$$k_1 \rightarrow x_j^{(1)} = \alpha_j^{(1)} e^{(\alpha+i\beta)t}$$

$$k_2 \rightarrow x_j^{(2)} = \alpha_j^{(2)} e^{(\alpha-i\beta)t}.$$

$\alpha_j^{(1)}, \alpha_j^{(2)}$ are determined from the system

$$\begin{pmatrix} \Delta(k) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\downarrow$ general soln.

$$e^{\alpha t}\left(\lambda_j^{(1)} \cos\beta t + \lambda_j^{(2)} \sin\beta t\right).$$

$$e^{\alpha t}\left(\bar{\lambda}_j^{(1)} \cos\beta t + \bar{\lambda}_j^{(2)} \sin\beta t\right)$$

$\lambda_j^{(1)}, \lambda_j^{(2)}, \bar{\lambda}_j^{(1)}, \bar{\lambda}_j^{(2)} \longrightarrow$ consts.

<u>Exercise</u>

$$\frac{dx_1}{dt} = -7x_1 + x_2$$

$$\frac{dx_2}{dt} = -2x_1 - 5x_2.$$

$k = -6 \pm i$