3.10.16

Example: Solve $(D^2-4)y = 2\sin\frac{x}{2}$.

Sol^n: put $\boxed{y=e^{mx}}$ in $\boxed{(D^2-4)y=0.}$

$m^2-4=0$ (auxiliary eqn).

$m=\pm 2$

$\boxed{C.F = c_1 e^{2x} + c_2 e^{-2x}.}$

linear DE.

$\boxed{f(D)y=0. \quad (1)}$

$\underline{y_1, y_2}$ two independent sol^n of (1).

$y = c_1 y_1 + c_2 y_2$ is also a sol^n of (1).

$$\begin{cases} f(D)y_1 = 0 \\ f(D)y_2 = 0. \end{cases}$$

$c_1, c_2 \to$ const

$$P.I. = \frac{1}{D^2-4}\, 2\sin\frac{x}{2}$$

$$= 2 \cdot \frac{1}{-\left(\frac{1}{2}\right)^2-4}\sin\frac{x}{2}$$

$$= 2 \cdot \frac{1}{-\frac{1}{4}-4}\sin\frac{x}{2}$$

$$= \frac{2 \times 4}{-17}\sin\frac{x}{2}$$

$$= -\frac{8}{17}\sin\frac{x}{2}$$

Cor. P.I.

- $\sin ax$ or $\cos ax \longrightarrow$ Replace D^2 by $-a^2$.

Example: Solve $(D^3+1)\, y = \sin 3x - \cos^2 \frac{x}{2}$

$$P.I. = \frac{1}{D^3+1}\left[\sin 3x - \frac{\cos x + 1}{2}\right].$$

$$= \frac{1}{D(-3^2)+1}\sin 3x - \frac{1}{2}\frac{1}{D(-1)^2+1}\cos x - \frac{1}{2}\cdot\frac{1}{D^3+1}e^{0x}$$

$$= \frac{1 \times (9D+1)}{(-9D+1) \times (9D+1)}\sin 3x - \frac{1}{2}\cdot\frac{1\,(1+D)}{(-D+1)(1+D)}\cos x. - \frac{1}{2}\cdot\frac{1}{0^3+1}e^{0x}.$$

$$= \frac{9D+1}{-81D^2+1}\sin 3x - \frac{1}{2}\frac{(D+1)}{-D^2+1}\cos x - \frac{1}{2}.$$

$$= \frac{9D+1}{-81(-9)+1}\sin 3x - \frac{1}{2}\cdot\frac{(D+1)}{-(-1)+1}\cos x - \frac{1}{2}.$$

$$= \frac{1}{730}(9D+1)\sin 3x - \frac{1}{4}(D+1)\cos x - \frac{1}{2}.$$

$$= \frac{1}{730}(9\cdot 3\cos 3x + \sin 3x) - \frac{1}{4}(-\sin x + \cos x) - \frac{1}{2}.$$

Example: Solve $(D^2+1)y = xe^{2x}$.

Soln:

$$P.I. = \frac{1}{D^2+1}\left(xe^{2x}\right)$$

$$= e^{2x}\cdot\frac{1}{(D+2)^2+1}\,x.$$

$$\boxed{= \frac{e^{2x}}{25}(5x-4).}$$

$$= e^{2x}\cdot\frac{1}{D^2+4D+4+1}\,x$$

$X = e^{ax}\cdot V(x)$

Replace D by $D+a$ to take out e^{ax} & then operate $\frac{1}{f(D+a)}$ on $V(x)$.

$$= e^{2x}\cdot\frac{1}{D^2+4D+5}\cdot x$$

$$P.I. = \frac{1}{f(D)}\,e^{ax}\cdot V(x).$$

$$= e^{2x}\cdot\frac{1}{5\left(1+\frac{D^2+4D}{5}\right)}\,x.$$

$$= e^{ax}\cdot\frac{1}{f(D+a)}\,V(x)$$

$$= \frac{e^{2x}}{5}\left[1 - \frac{D^2+4D}{5} + \left(\frac{D^2+4D}{5}\right)^2 - \cdots\right]x.$$

$$= \frac{e^{2x}}{5}\left[x - \frac{4}{5}\right] = \frac{e^{2x}}{25}(5x-4).$$

$Dx = 1$

$D^2x = 0$

$\vdots$

- $X = x^m$, m +ve integer, $\boxed{m > 0.}$

$\frac{1}{f(D)} x^m =$ go for binomial expansion of $f(D)$.

Example: $(D^2-1)y = x^2 \cos x$.

P.I. $=$ $\cancel{Re \frac{1}{D^2-1} x^2 e^{ix}}$

$$= Re \left[\frac{1}{(D^2-1)} x^2 e^{ix} \right].$$

$$= Re \left[e^{ix} \cdot \frac{1}{(D+i)^2 - 1} x^2 \right].$$

$$= Re \left[e^{ix} \cdot \frac{1}{D^2 + 2iD - 1 - 1} x^2 \right]$$

$$= Re \left[e^{ix} \cdot \frac{1}{D^2 + 2iD - 2} x^2 \right].$$

$$= Re \left[e^{ix} \frac{1}{-2\left(1 - \frac{D^2+2iD}{2}\right)} x^2 \right].$$

$$= \mathrm{Re}\left[\frac{e^{ix}}{-2}\left\{1 + \frac{D^2+2iD}{2} + \left(\frac{D^2+2iD}{2}\right)^2 + \cdots\right\}x^2\right]$$

$$= \mathrm{Re}\left[\frac{e^{ix}}{-2}\left\{x^2 + \frac{1}{2}(2+2i\cdot 2x) + \frac{1}{4}4i^2\cdot 2\right\}\right].$$

$$= \mathrm{Re}\left[\frac{e^{ix}}{-2}\left\{x^2 + 2ix + 1 - 2\right\}\right]$$

$$= \frac{1}{2}(1-x^2)\cos x + x\sin x.$$

$D(x^2) = 2x$
$D^2(x^2) = 2$
$D^3(x^2) = 0$
$\vdots$

<u>Exercise</u> find a P.I. for

i) $(D^2+5D+6)y = e^{-2x}\sin 2x$.
normal.

ii) $(D^2+9)y = x\sin x$.
$\mathrm{Im}\left[\frac{1}{D^2+9}xe^{ix}\right]$

iii) $\cancel{D^2+a^2}$ $(D^2+1)y = \sec^2 x$.
use partial fraction

$$\text{P.I.} = \frac{1}{D^2+1}\sec^2 x$$

$$= \frac{1}{(D+i)(D-i)} \sec^2 x .$$

$$= \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \sec^2 x .$$

$$\frac{1}{D-i} \sec^2 x = y .$$

$(D-i)\, y = \sec^2 x \rightarrow$ linear in y

$$y = e^{ix} \int e^{-ix} \sec^2 x \, dx .$$

$$= e^{ix} \int \frac{(\cos x - i \sin x)}{\cos^2 x} dx .$$

$$= e^{ix} \int (\underline{\sec x} - i \tan x \sec x) dx$$

$$= e^{ix} \int \frac{\sec x . (\sec x + \tan x)}{(\sec x + \tan x)}$$

$$- i e^{ix} \int d(\sec x) dx .$$

$$= e^{ix} [\log(\sec x + \tan x) - i \sec x] .$$

Linear equ^ns. with variable co-efficients

consider only homogeneous equ^n

Homogeneous linear equ^ns. ($\boxed{\text{Cauchy's linear equ}^n}$)

$$① — x^n \frac{d^n y}{dx^n} + p_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + p_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots$$

$$\cdots + p_{n-1} x \frac{dy}{dx} + p_n y = X.$$

where $p_1, p_2, \ldots, p_n$ are constants; & X in a fu^n. of x.

- $X = 0 \rightarrow$ Cauchy-Euler's equ^n. or $\boxed{\text{Euler's equ}^n}$

Transforming ① into equ^n. with constant coefficient.

put $\boxed{x = e^z.}$ i.e. $z = \log x$.

~~Rewrit~~

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}.$$

$$D \equiv \frac{d}{dz}.$$

$$x\frac{dy}{dx} = Dy$$

$$x^2\frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y.$$

lly $$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y.$$

$$\vdots$$

$$\underline{x^n\frac{d^ny}{dx^n}} = \underline{D(D-1)\cdots(D-n+1)y.}$$

Solve

Example: $$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x.$$

Soln. put $x = e^z$ i.e. $z = \log x$.

$$[D(D-1) - D + 1]\,y = 2z., \quad D \equiv \frac{d}{dz}.$$

$m^2 - 2m + 1 = 0$

$m = 1, 1.$

$$C.F = (C_1 + C_2 z)e^z.$$

$$= (C_1 + C_2\log x)x.$$

$$P.I. = \frac{1}{(D-1)^2}\,2z. \quad = 2[\log x + 2].$$

(verify).

Example: Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$. —(1).

Sol^n: put $x = e^z$. i.e. $z = \log x$

Then (1) reduces to

$$[D(D-1) - 2D - 4]\,y = e^{4z} \quad \text{—(2).}$$

To find C.F.

put $y = e^{mx}$ in the eq^n.

$$[D(D-1) - 2D - 4] = 0,$$

$$m^2 - m - 2m - 4 = 0 \qquad D \equiv \frac{d}{dz}.$$

$$m^2 - 3m - 4 = 0.$$

$$m^2 - 4m + m - 4 = 0.$$

$$(m-4)(m+1) = 0$$

$$m = 4, -1.$$

$$\text{C.F} = C_1 e^{4z} + C_2 e^{-z}, \quad C_1, C_2 \text{ consts.}$$

$$= C_1 x^4 + \frac{C_2}{x} \quad \left(\text{as } x = e^z\right).$$

$$P.I. = \frac{1}{(D-4)(D+1)} e^{4z}.$$

$$= e^{4z} \cdot \frac{1}{4+1} \cdot \frac{1}{D-4} \cdot 1$$

$$= \frac{e^{4z}}{5} \cdot z$$

$$= \frac{x^4}{5} \log x.$$

The complete integral is

$$y = C.F. + P.I.$$

$$= c_1 x^4 + \frac{c_2}{x} + \frac{x^4}{5} \log x, \quad c_1, c_2 \text{ two arb. consts.}$$

<u>Example:</u> Solve

$$(x^2 D^2 + xD + 1) y = \log x . \sin \log x \quad \text{— (1)}$$

$\downarrow$ put $x = e^z$ i.e. $z = \log x$.

$D_1 \equiv \frac{d}{dz}$.

$$[D_1(D_1 - \cancel{1}) + \cancel{D_1} + 1] y = z \sin z. \quad \text{— (2)}$$

$\downarrow$ C.F.

put $y = m x$ in (2).

$m = \pm i$

$$C.F = c_1 \cos z + c_2 \sin z.$$

$$= c_1 \cos(\log x) + c_2 \sin(\log x)$$

$$P.I. = \frac{1}{D_1^2+1} z \sin z.$$

$$= I_m \left[\frac{1}{D_1^2+1} z e^{iz} \right]$$

$$= \frac{1}{4} \log x \sin \log x - \frac{1}{4} (\log x)^2 \cos \log x$$

(check).

Example: find the value of λ for which all solutions of

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} - \lambda y = 0$$ tends to zero as $x \to \infty$.

Ans. $-1 \leq \lambda < 0$ or $\lambda < -1$.

-1 $\quad$ 0

Equations reducible to the homogeneous linear form.

(1) —
$$\boxed{(a+bx)^n \frac{d^n y}{dx^n} + p_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_{n-1} (a+bx) \frac{dy}{dx} + p_n y = F(x)}$$

$p_1, p_2, \ldots, p_n \rightarrow$ consts.

→ Legendre linear eqn.

↓

can be reduced to

Cauchy ~~[illegible]~~ eqn.

put $z = a+bx$

Then (1) reduces to

(2) —
$$z^n \frac{d^n y}{dz^n} + \frac{p_1}{b} z^{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \cdots + \frac{p_n y}{b^n} = \frac{F\left(\frac{z-a}{b}\right)}{b^n}$$

Example: Solve

$$\left[(1+2x)^2 D^2 - 6(1+2x)D + 16\right] y = 8(1+2x)^2$$

Sol^n.

put $1+2x = z$.

$\downarrow$

Convert it into Cauchy's equ^n. homogeneous equ^n.

$\downarrow$

$z = e^t$. i.e. $t = \log z$.

$\downarrow$

Convert it into linear equ^n. with constant coefficient.

$$\left(z^2 D_1^2 - 3z D_1 + 4\right) y = 2z^2, \quad D_1 \equiv \frac{d}{dz}.$$

$\downarrow$

$$\left(D_2^2 - 4D_2 + 4\right) y = 2e^{2t}, \quad D_2 \equiv \frac{d}{dt}$$

$\downarrow$

final sol^n. $\boxed{y = \text{in terms of } x}$

- linear ODE.

  - Short methods to find P.I.
  - Not all problems solvable by these short method.
  - $\boxed{\text{Variation of parameters.}}$
  - Numerical methods. (will not discuss).

linear equ^ns. with constant co-efficient

- $(D^n + p_1 D^{n-1} + \cdots + p_n) y = X$, — 1

  $p_1, p_2, \ldots, p_n \to$ consts.

  $X \to$ fn. of x.

  $D \to \cancel{\frac{D}{dx}}\ \frac{d}{dx}$.

- $C.F = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$.
- To find a particular integral

let the P.I. be

$$y = L_1(x)\, y_1(x) + L_2(x)\, y_2(x) + \cdots + L_n(x)\, y_n(x) \quad \text{--- (2)}.$$

$\rightarrow$ obtained by replacing $c_1, c_2, \ldots, c_n$ in C.F of (1) by $L_1(x), L_2(x), \ldots, L_n(x)$ resp'ly.

- Determine $L_1(x), L_2(x), \ldots, L_n(x)$ so that (2) satisfies (1).

<u>Example</u>: Solve $(D^2 - 2D)y = e^x \sin x$ --- (1) using variation of parameters.

<u>Sol</u>

$m^2 - 2m = 0$

$m = 0, 2$

$$C.F = c_1 e^{0x} + c_2 e^{2x}$$
$$= c_1 + c_2 e^{2x}.$$

let P.I. $y = L_1(x) + L_2(x) e^{2x}$
$$= L_1 + L_2 e^{2x}.$$

$$Dy = \left(L_1' + L_2' e^{2x}\right) + 2L_2 e^{2x}.$$

Set $\quad L_1' + L_2' e^{2x} = 0 \quad \text{———} \quad (2).$

$$\left.\begin{aligned} \therefore\ Dy &= 2L_2 e^{2x}. \\ D^2y &= 2L_2' e^{2x} + 4L_2 e^{2x} \end{aligned}\right\} \text{—} (3).$$

(1), (3) $\Rightarrow$

$$2L_2' e^{2x} + \cancel{4L_2 e^{2x}} - \cancel{4L_2 e^{2x}} = e^{x}\sin x.$$

$$\boxed{L_2' = \frac{e^{-x}\sin x}{2}.}$$

(2) $\Rightarrow$ $\boxed{L_1' = -L_2' e^{2x} = -\frac{e^{x}\sin x}{2}.}$

$$L_1 = -\frac{1}{4}e^{x}\left(-\cos x + \sin x\right)$$

$$L_2 = -\frac{1}{4}e^{-x}\left(\cos x + \sin x\right)$$

$$\text{P.I.} = L_1 + L_2 e^{2x}$$

$$= -\frac{1}{2}e^{x}\sin x.$$

Method of variation of parameters.

$$(D^2 + PD + Q)y = R.$$

$P, Q \to$ consts.

$R \to R(x).$

C.F. $= C_1 u + C_2 v.$, u, v are two linearly independent sol^{ns}. of $(D^2 + PD + Q)y = 0$

$$(D^2 + PD + Q)u = 0$$

$$(D^2 + PD + Q)v = 0.$$

let P.I. $y = L_1(x)u + L_2(x)v$

$= L_1 u + L_2 v.$

$Dy = L_1' u + L_2' v + L_1 u' + L_2 v'.$

$$\boxed{L_1' u + L_2' v = 0} \text{ — } (1)$$

$= L_1 u' + L_2 v'.$

$D^2 y = L_1 u'' + L_2 v'' + L_1' u' + L_2' v'.$

$\therefore$ given DE

$$(D^2 + PD + Q)\, y = R$$

$$\Rightarrow \underline{L_1} u'' + L_2 v'' + L_1' u' + L_2' v' + P(\underline{L_1 u'} + L_2 v') + Q(\underline{L_1 u} + L_2 v) = R$$

$$\Rightarrow \boxed{L_1' u' + L_2' v' = R.} \quad (2)$$

u, v is a soln. of
$(D^2 + PD + Q)\, y = 0.$
$\Downarrow$
$u'' + Pu' + Qu = 0.$
&
$v'' + Pv' + Qv = 0.$

$$L_1' u + L_2' v = 0.$$
$$L_1' u' + L_2' v' - R = 0.$$

$$\boxed{W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}$$

$$\frac{L_1'}{-Rv} = \frac{L_2'}{Ru} = \frac{1}{uv' - u'v}$$

Wronskian of u, v.

$$\Rightarrow \left\{ \begin{aligned} L_1' &= \frac{-Rv}{uv' - u'v} \\ L_2' &= \frac{Ru}{uv' - u'v}. \end{aligned} \right.$$

$$\Downarrow$$

$$L_1 = -\int \frac{Rv}{W}\,dx$$

$$L_2 = -\int \frac{Ru}{W}\,dx.$$

<u>Third order ODE.</u>

$$(D^3 + PD^2 + QD + R)\,y = S.$$

$P, Q, R \rightarrow$ consts.

$S \rightarrow s(x)$.

$u, v, w \rightarrow$ 3 ind. soln. of $(D^3 + PD^2 + QD + R)\,y = 0$.

$$C.F = C_1 u + C_2 v + C_3 w.$$

$$P.I = L_1 u + L_2 v + L_3 w,$$ L_1, L_2, L_3 fun. of x.

Wronskian of u, v, w

$$= \begin{vmatrix} u & v & w \\ u' & v' & w' \\ u'' & v'' & w'' \end{vmatrix}$$

$$L_1' = \frac{S}{W}\begin{vmatrix} v & w \\ v' & w' \end{vmatrix}, \quad L_2' = -\frac{S}{W}\begin{vmatrix} u & w \\ u' & w' \end{vmatrix}$$

$$L_3' = \frac{S}{W}\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}.$$

Solving $\rightarrow L_1, L_2, L_2$.

1. $\frac{1}{D^2} x = y \longrightarrow \int(dx) \int x\, dx$

$\boxed{D^2 y = x}$

2. $\frac{1}{(D+2)^2} x = y$

$(D+2)^2 y = x.$

$(D+2) \underline{(D+2) y} = x.$

3. $X = e^{ax}$

$f(D) = (D-a)^l \underline{\underline{\phi(D)}}$

let $m_1 = m_2 = \dots = m_l = a.$

$f(D) = (D-m_1)(D-m_2)\dots(D-m_n)$

$\frac{1}{f(D)} X = e^{-m_1 x} \int e^{(m_2-m_1)x} \int \dots \int X (dx)^n.$

$\Rightarrow$ ~~$f(D)$~~ ~~$f(D) = e^{ax}$~~

$\frac{x^l}{l!}$ ← gives l such integration equals ← when $m_2 = m_1$, the integration gives x.