28.9.16

ODE linear ~~•~~ of ~~•~~ order n

with constant co-efficients.

① — $f(D)y = \cancel{\phi}\, X$

$$f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_{n-1} D + P_n$$

$P_1, P_2, \ldots, P_n \rightarrow$ constants.

$X \rightarrow$ fn. of x.

$$D \equiv \frac{d}{dx}.$$

General soln / complete integral for ①.

$$y = \underline{\underline{C.F}} + P.I.$$

$$\boxed{f(D)y = 0.} \longrightarrow C.F.$$

$$\text{P.I.} = \frac{1}{f(D)} X .$$

I> $X = e^{ax}$, a const.

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax} \quad \left(\text{Replace } D \text{ by } a\right)$$

provided $f(a) \neq 0$.

$\boxed{\text{if } f(a) = 0}$

$$f(D) = (D-a)^{l} \, \underline{\phi(D)} .$$

$\phi(a) \neq 0$.

$$\boxed{\begin{aligned} \text{P.I.} &= \frac{1}{\phi(a)} e^{ax} \frac{1}{(D-a)^{l}} \cdot 1 \\ &= \frac{1}{\phi(a)} e^{ax} \frac{x^{l}}{l!} \end{aligned}}$$

Example: find a P.I. for the DE

$$(D^3+1)\,y = 3 + e^{-x} + 5e^{2x}.$$

Sol^n.

$$\text{P.I.} = \frac{1}{D^3+1}\left(3 + e^{-x} + 5e^{2x}\right)$$

$3 \to \times e^{0x}$

$$= \frac{1}{0^3+1} \times 3e^{0} + 5 \cdot \frac{1}{2^3+1} e^{2x}$$

$$+ \frac{1}{(D+1)(D^2-D+1)} e^{-x}.$$

$$= 3 + \frac{5}{9} e^{2x} + \frac{1}{(-1)^2-(-1)+1} e^{-x} \cdot \frac{1}{D+1} 1$$

$$= 3 + \frac{5}{9} e^{2x} + \frac{1}{3} x e^{-x}.$$

$$\frac{1}{D} 1 = \int 1\,dx = x.$$

$D^n y \to Dy$

$\frac{1}{D^n} y \to \frac{1}{D} y.$

II) $X = x^m$, m +ve integer, $m > 0$.

$$P.I. = \frac{1}{f(D)} x^m .$$

$$P_{l-1} D^{l-1}$$

$$f(D) = D^n + P_1 D^{n-1} + \dots + P_l D^l$$

$l \rightarrow$ lowest degree of D

$$\cancel{P.I = \frac{1}{P_l D^l}\left[1 + \frac{D^n + P_1 D^{n-1} + \dots + P_{l-1} D^{l-1}}{P_l D^l}\right.}$$

$$P.I = \frac{1}{P_l D^l} \underbrace{\left[1 + \frac{D^n + P_1 D^{n-1} + \dots + P_{l-1} D^{l-1}}{P_l D^l}\right]^{-1}}_{\text{expand.}} x^m .$$

Solve.

Example:

$$(D^3+3D^2+2D)\,y = x^2.$$

Soln.

$$y = e^{mx}$$

auxiliary eqn.

$$m^3+3m^2+2m=0.$$

$$m = 0, -1, -2.$$

$$C.F = C_1 e^{0x} + C_2 e^{-1x} + C_3 e^{-2x},$$

C_1, C_2, C_3 arb. const.

$$= C_1 + C_2 e^{-x} + C_3 e^{-2x}.$$

$$P.I. = \frac{1}{D^3+3D^2+\underline{\underline{2D}}}\, x^2.$$

$$= \frac{1}{2D\left[1+\frac{D^2+3D}{2}\right]}\, x^2.$$

$$= \frac{1}{2D}\left[1-\frac{D^2+3D}{2}+\left(\frac{D^2+3D}{2}\right)^2-\cdots\right]x$$

$$= \frac{1}{2D}\left[x^2 - \frac{1}{2}\left\{3 \cdot 2x + 2\right\} + \frac{1}{4}\left\{9 \cdot 2\right\}\right]$$

$$= \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right].$$

$D^4 + 6D^3 + 9D^2$

$D(x^2) = 2x$

$D^2(x^2) = 2$

Complete integral is

$$y = \cancel{P.I.}\ C.F + P.I.$$

III > . $X = \cos ax$, a const.

A> $P.I. = \frac{1}{f(-a^2)} \cos ax.$

Replace D^2 by $-a^2$.

if $f(-a^2) = 0$

$$P.I = Re\left[\frac{1}{f(D)} e^{iax}\right]$$

B> $X = \sin ax$, a const.

$$P.I = \frac{1}{f(-a^2)} \sin ax$$

if $f(-a^2) = 0$

$$P.I = Im\left[\frac{1}{f(D)} e^{iax}\right].$$

Example: $(D^3+D^2-D-1)y = \cos 2x$

$P.I = ?$

$\underline{\underline{Sol^n}}$ $P.I = \dfrac{1}{D^3+D^2-D-1} \cos 2x.$

$= \dfrac{1}{-4D-4-D-1} \cos 2x.$

(Replacing D^2 by $-2^2 = -4.$)

$= \dfrac{1}{-5D-5} \cos 2x$

$= -\dfrac{1}{5} \cdot \dfrac{1}{D+1} \cos 2x.$

$= -\dfrac{1}{5} \dfrac{D-1}{D^2-1} \cos 2x.$

$= -\dfrac{1}{5} \dfrac{(D-1)}{-4-1} \cos 2x$

$= \dfrac{1}{25} (D-1) \cos 2x.$

$$= \frac{1}{25}\left[-2\sin 2x - \cos 2x\right].$$

Example: $(D^2+a^2)y = \cos ax.$

$$P.I = \frac{1}{D^2+a^2}\cos ax.$$

$$= Re\left[\frac{1}{D^2+a^2}e^{iax}\right]$$

$$= Re\left[\frac{1}{(D+ia)(D-ia)}e^{iax}\right]$$

$$= Re\left[\frac{1}{2ia}e^{iax}\cdot\frac{1}{D-ia}\cdot 1\right]$$

$$= Re\left[\frac{1}{2ia}e^{iax}\cdot x\right]$$

$$= Re\left[\frac{1}{2ia}x(\cos ax + i\sin ax)\right]$$

$$= \frac{x\sin ax}{2a}.$$

IV) $P.I = \dfrac{1}{f(D)} \underline{e^{ax} V}$, V is any fu^n of x, V is not a const.

Replace D by D+a.

$$P.I = e^{ax} \cdot \frac{1}{f(D+a)} V.$$

Example: $(D^2 - 2D + 1)y = \underline{x e^x \sin}$

Short methods

I.) e^{ax}, a any const. $\longrightarrow$ Replace D by a

II) x^m, m +ve int. $\longrightarrow$ Binomial expansion

III) $\cos ax$ or $\sin ax$ $\longrightarrow$ Replace D^2 by $-a^2$

IV) $e^{ax} V(x)$ $\longrightarrow$ Replace D by D+a to take out e^{ax} & then operate $\dfrac{1}{f(D+a)}$ on $V(x)$

$$\text{P.I.} = \frac{1}{D^2-2D+1}\, x e^x \sin x$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 2(D+1)+1}\, x \sin x$$

$$= e^x \cdot \frac{1}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - \cancel{2} + \cancel{1}}\, x \sin x$$

$$= e^x \cdot \frac{1}{D^2}\, x \sin x .$$

$$= e^x \cdot \frac{1}{D} \int x \sin x \, dx$$

$$= e^x \frac{1}{D} \left(-x \cos x + \sin x\right).$$

$$= e^x \int \left(-x \cos x + \sin x\right) dx$$

$$= -e^x \left(x \sin x + 2\cos x\right)$$

<u>(check)</u>

<u>Example:</u> $(D^2+a^2)y = \sec ax.$

$$P.I = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D+ia)(D-ia)}\sec ax.$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax.$$

$$\frac{1}{D+ia}\sec ax = u.$$

$$(D+ia)u = \sec ax.$$

$$\frac{du}{dx} + iau = \sec ax$$ linear in u.

$$I.F. = e^{iax}.$$

$$d\left(e^{iax}u\right) = \sec ax\, e^{iax}\,dx$$
$$= \sec ax\,(\cos ax + i\sin ax)\,dx$$
$$= (1 + i\tan ax)\,dx$$

$$u = e^{-iax}\left[x - \frac{i}{a}\log(\cos ax)\right]$$

$$\frac{1}{D - ia}\sec ax = v.$$

$$v = e^{iax}\left[x + \frac{i}{a}\log(\cos ax)\right]$$

$$P.I = \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax\, \cancel{\log(\cos}\, \log(\cos ax)$$

$$\frac{1}{2ia}[v - u]$$

Ex. $(D^2 + a^2)y = X, \quad X = \begin{cases} \sec ax \\ \tan ax \\ \cot ax \end{cases}$

General method to find P.I.

(Two approaches).

I) $\dfrac{1}{f(D)}$ may be decomposed into its partial fraction.

$$P.I = \frac{1}{f(D)} X$$

$$= \left[\frac{N_1}{D-m_1} + \frac{N_2}{D-m_2} + \cdots + \frac{N_n}{D-m_n} \right] X$$

$$\boxed{f(D) = D^n + P_1 D^{n-1} + \cdots + P_{n-1} D + P_n}$$

$P_1, P_2, \cdots, P_n$ are all consts.

II > $\frac{1}{f(D)}$ may be factored.

$$P.I = \frac{1}{f(D)} X$$

$$= \frac{1}{(D-m_1)} \cdot \frac{1}{(D-m_2)} \cdots \frac{1}{(D-m_n)} X.$$

$$\boxed{\frac{1}{D-m_n} X = y}$$

$$(D-m_n) y = X$$

↳ linear 1st. order in y

$$I.F = e^{-m_n x}$$

$$d\left(e^{-m_n x} y\right) = X e^{-m_n x}$$

$$y = e^{m_n x} \int X e^{-m_n x} \, dx.$$

↳ no const. as P.I.

Apply $\frac{1}{D-m_{n-1}}$ on this y & so on.

finally,

$$P.I. = e^{m_1 x}\int e^{(m_2-m_1)x}\int \cdots \int X e^{-m_n x}(dx)^n.$$

use to derive all short methods for P.I.

Example: $X = e^{ax}$.

$$P.I = \frac{1}{f(a)}e^{ax}.$$

Check

$$\frac{1}{D}x = \int x\,dx$$

$$\frac{1}{D+a}x = y$$

$(D+a)y = x$

$y = ?$