26.9.16

1st. order 1st. degree Ordinary DE.

- Exact DE.
- Finding IF.
- linear DE. $\boxed{\frac{dy}{dx} + Py = Q}$ → linear in y

$P, Q \rightarrow$ fn. of x.

- Equations reducible to the linear eqn.

(Bernoulli's eqn.).

$$\boxed{\frac{dy}{dx} + Py = Q y^n}$$

$P = P(x)$
$Q = Q(x)$.

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{1}{y^{n-1}} P = Q.$$

let $v = \frac{1}{y^{n-1}} = y^{1-n}$

$$\therefore \frac{dv}{dx} = (1-n)\, y^{1-n-1} \frac{dy}{dx}$$

$$= (1-n)\frac{1}{y^n}\frac{dy}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}.$$

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q.$$

$$\boxed{\frac{dv}{dx} + \underbrace{(1-n)P}_{\text{fn. of } x}\, v = \underbrace{Q}_{\text{fn. of } x}.}$$

linear DE

in v.

Example: Solve $\frac{dy}{dx} + \frac{xy}{1-x^2} = x\,y^{1/2}$.

$$\frac{1}{y^{\frac{1}{2}}}\frac{dy}{dx} + \frac{x}{1-x^2}\, y^{\frac{1}{2}} = x.$$

put $\boxed{v = y^{\frac{1}{2}}}$

get linear DE in v.

general sol^n is

$$y^{1/2} = c\,(1-x^2)^{1/4} - \frac{1}{3}(1-x^2).$$

(check).

DE of 1st. order, higher degree

- denote $\frac{dy}{dx}$ as p.
- $\boxed{f(x, y, p) = 0.}$ where

$$f(x,y,p) = p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n .$$

- $P_1, P_2, \ldots, P_n$ are fn^n! of x, y.

I> Equ^n! solvable for p.

$$f(x,y,p) = (p - R_1)(p - R_2) \cdots (p - R_n) = 0$$

$R_1, R_2, \ldots, R_n \rightarrow$ fn^ns! of x, y.

$\rightarrow$ $p = R_1, R_2, \ldots, R_n$

$\rightarrow$ sol^n! easy to get.

Example: $4y^2p^2 + 2pxy(3x+1) + 3x^3 = 0.$

$$(2yp + 3x^2)(2yp + x) = 0.$$

$2y\frac{dy}{dx} + 3x^2 = 0 \Rightarrow 2 \cdot \frac{y^2}{2} + 3 \cdot \frac{x^3}{3} = c_1$

$\Rightarrow y^2 + x^3 = c_1.$

or

$$2y\frac{dy}{dx} + x = 0 \Rightarrow 2\frac{y^2}{2} + \frac{x^2}{2} = \frac{c_2}{2}$$

$$\Rightarrow y^2 + \frac{x^2}{2} = c_2 .$$

II > Equations solvable for y.

$$f(x, y, p) = 0 \Rightarrow y = F(x, p) \quad \text{--- (1)}$$

$\Big\downarrow$ w.r.to x

$$p = \phi\left(x, p, \frac{dp}{dx}\right).$$

$\Big\downarrow$ try to deduce a relation

$$\psi(x, p, c) = 0, \quad c \text{ const.} \quad \text{--- (2)}.$$

eliminate p from (1) & (2).

$\rightarrow$ general solⁿ.

Example: Solve $4y = x^2 + p^2$ —— (1).

Sol^n. Differentiating (1) both sides w.r.to x

$$4p = 2x + 2p\frac{dp}{dx}.$$

$$\frac{dp}{dx} = \frac{2p - x}{p}.$$ $\boxed{\text{put } p = vx.}$

$$\frac{dp}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2v-1}{v}$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v = \frac{2v-1-v^2}{v} = -\frac{(v-1)^2}{v}$$

$$\frac{(v-1+1)}{(v-1)^2}dv = -\frac{dx}{x}.$$

$$\left[\frac{1}{v-1} + \frac{1}{(v-1)^2}\right]dv = -\frac{dx}{x}$$

$$\log|v-1| - \frac{1}{v-1} = -\log x + c.$$

$$\Rightarrow \log|p-x| - \frac{x}{p-x} = c \text{ —— (2)}.$$

General sol^n. is obtained by eliminating p from (1) & (2).

III > Equations solvable for x

$$f(x,y,p)=0 \Rightarrow x = F(y,p) \quad \text{--- (1)}$$

$\downarrow$ w.r.to y

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right).$$

$\downarrow$ try to get a relation of the form

$$\psi(p, y, c) = 0 \quad \text{--- (2)}$$

eliminate p from (1) & (2) $\rightarrow$ gives the gen. sol^n.

Example: Solve $x = y + p^2$.

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

$$\frac{dp}{dy} = \frac{\frac{1}{p} - 1}{2p} = \frac{1-p}{2p^2}.$$

$$\frac{p^2}{1-p}\,dp = \frac{dy}{2}$$

$\hookrightarrow$ Solve it. $\left[-p-1-\frac{1}{-1+p}\right]$

Complete it.

IV> **Equations that do not contain x.**

$f(y, p) = 0$
- → Solvable for p — case I.
- → Solvable for y — case II.

Equations that do not contain y

$f(x, p) = 0$
- → Solvable for p — case I.
- → Solvable for x — case III.

Example: Solve the DE

$$x(1+p^2) = 1.$$

$$x = \frac{1}{1+p^2}$$

↓ diff. w.r.to y

$$\frac{1}{p} = -\frac{1}{(1+p^2)^2}\, 2p\, \frac{dp}{dy}.$$

$$\frac{dy}{dx} = p = \sqrt{\frac{1-x}{x}}$$

Ans. The gen. sol^n. is

$$\sqrt{\frac{1-x}{x}} \cdot x - \tan^{-1}\sqrt{\frac{1-x}{x}} = y + c,$$

c const.

V Equations homogeneous in x & y.

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0, \qquad \frac{dy}{dx} \to p$$

Solvable for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{F_1(x,y)}{F_2(x,y)}$$

F_1, F_2 hom. in x, y of same degree

Solvable for $\frac{y}{x}$.

$$\boxed{y = x\, f(p).}$$

Case II.

Example:

$$y = yp^2 + 2px \quad \text{—— (1)}.$$

$$\frac{y}{x}\left[1 - p^2\right] = 2p.$$

$$\frac{y}{x} = \frac{2p}{1-p^2}$$

$y = x \cdot \frac{2p}{1-p^2}$ Apply case II.

VI > Equations of the 1st. degree in x, y.

- **Clairaut's eqn.**

$$y = px + f(p) \quad \text{--- (1)}$$

general soln. is

$$\boxed{y = cx + f(c)}, \quad c \text{ is a const.}$$

- Why?

→ Diff. w.r.t. x

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}.$$

$$\frac{dp}{dx}\left[x + f'(p)\right] = 0 . \qquad \to \frac{\partial F}{\partial p}.$$

→ $p = c$, c const. --- (2).

- $x + f'(p) = 0$ will not give a general soln.

→ Singular soln.

- $F(x,y,p)=0.$; $\dfrac{\partial F}{\partial p}=0.$ $\Big\}$ eliminate p.

↓

p-discriminant.

$E T^2 C$.

E → Envelop → singular sol^n.

T → tac

C → cusp.

Clairaut's form.

$$y = px + f(p)$$

$$F(x,y,p) = y - px - f(p) = 0.$$

$$\frac{\partial F}{\partial p} = \boxed{-x - f'(p) = 0}$$

- $\boxed{y = x f_1(p) + f_2(p).}$

1st. degree in x, y.

in particular, $f_1(p) = p$

↓

Clairaut's form.

w.r.to x differentiate

$$p = f_1(p) + x\, f_1'(p)\frac{dp}{dx} + f_2'(p)\frac{dp}{dx}$$

or $$\frac{dp}{dx}\left[x f_1'(p) + f_2'(p)\right] = p - f_1(p)$$

$$\frac{dx}{dp} = \frac{x f_1'(p) + f_2'(p)}{p - f_1(p)}.$$

$$\frac{dx}{dp} - \frac{f_1'(p)}{p - f_1(p)}\, x = \frac{f_2'(p)}{p - f_1(p)}.$$

$\hookrightarrow$ linear DE in x.

- <u>Some eqn. are reducible to Clairaut's form</u>

<u>Example:</u> $x^2(y - px) = y p^2$.

<u>Soln.</u>

$x^2 = u$, $y^2 = v$.

$2x\,dx = du$, $2y\,dy = dv$.

$$\frac{y}{x}p = \frac{dv}{du}.$$

$$\Rightarrow p = \frac{x}{y}\frac{dv}{du}.$$

$$x^2\left(y - \frac{x}{y}\cdot x\frac{dv}{du}\right) = \cancel{y}\cdot\frac{x^2}{\cancel{y^2}_{y}}\left(\frac{dv}{du}\right)^2$$

$$\cancel{x^2}\left(y^2 - x^2\frac{dv}{du}\right) = \cancel{x^2}\left(\frac{dv}{du}\right)^2$$

$$\left(v - u\frac{dv}{du}\right) = \left(\frac{dv}{du}\right)^2$$

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2.$$

$\hookrightarrow$ clairaut's form.

general soln. is

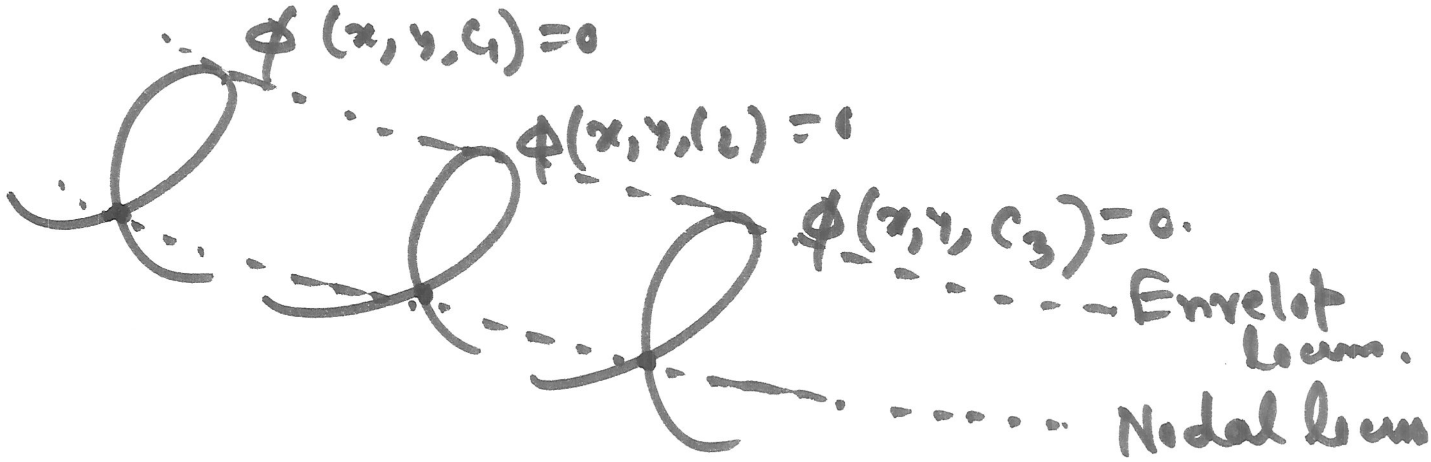
$$v = uc + c^2.$$

$$\boxed{y^2 = cx^2 + c^2}, \quad c \text{ const.}$$

- $f(x,y,p)=0$. (given DE).
- $\phi(x,y,c)=0$ (general sol^n.).

→ different curves for different values of c.

(infinitely many such curves).



(locus of double pts.)

- E (envelop locus) → curve that is tangent to each member of the family $\phi(x,y,c)=0$ at some pt.
- N (Nodal locus).
- C (cuspidal locus) → locus of cusp pts. where direction of motion is exactly reversed
- T (Tac locus)

- c-discriminant

$$\left.\begin{aligned} \phi(x, y, c) &= 0 \\ \frac{\partial \phi}{\partial c} &= 0 \end{aligned}\right\} \quad \text{eliminate } c$$

~~ET²C.~~

EN²C³.

also satisfies the given DE.

singular soln.

- p-discriminant

$$\left.\begin{aligned} f(x, y, p) &= 0. \\ \frac{\partial f}{\partial p} &= 0. \end{aligned}\right\} \quad \text{eliminate } p.$$

ET²C.

~~ET~~

~~EN²C³.~~

Example: Find the Nodal locus, Envelope, Tac locus, Cusp locus if any of the DE

$$xp^2 - (x-a)^2 = 0. \quad \text{— (1).}$$

$$p = \frac{x-a}{\sqrt{x}}$$

general soln

$$y + c = \frac{2}{3}\sqrt{x}\,(x - 3a).$$

$$\frac{9}{4}(y+c)^2 = x\,(x-3a)^2.$$

$$\phi(x,y,c) = \frac{9}{4}(y+c)^2 - x\,(x-3a)^2 = 0 \quad \text{— (2)}$$

c-disc

$$\frac{\partial \phi}{\partial c} = 0 = \frac{9}{4}\,2\,(y+c)$$

$$\Rightarrow y = -c.$$

eliminate c.

$$\boxed{x\,(x-3a)^2 = 0} \qquad E N^2 \underset{\times}{C^3}.$$

p-disc

(1) $\Rightarrow \quad f(x,y,p) = xp^2 - (x-a)^2 = 0.$

$$\frac{\partial f}{\partial p} = 2px = 0.$$

$$\Rightarrow xp = 0.$$

$$(xp)^2 - x(x-a)^2 = 0.$$

$$\boxed{x(x-a)^2 = 0} \qquad ET^2C.$$

- $x = 0 \rightarrow$ Envelop as it is in both c-disc, p-disc. & also satisfies the given DE

- $x = 3a \rightarrow$ Nodal locus.

- $x = a \rightarrow$ Tac locus.

- No cuspidal locus.

Exercise $\quad f(x,y,p) = p^2 + 2xp - y = 0.$

E? N? C? T?
if any.

Example: (Exception) (both Envelop & the cusp locus coincide.)

$8ap^3 = 27y.$

Check it

General linear DE with constant co-efficients.

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X$$

or

$$\boxed{\left[D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right] y = X,} \quad (1).$$

$P_1, P_2, \ldots, P_n, X \to$ either constants or fn's of x.

$D \equiv \frac{d}{dx}.$

$\frac{dy}{dx} + Py = Q.$

$\underline{n=1}$

• Consider $P_1, P_2, \ldots, P_n$ → all constants.

$$\boxed{f(D)\,y = X} \quad \text{——— (2)}.$$

$$f(D) = D^n + P_1 D^{n-1} + \cdots + P_n.$$

general Sol^n. involves two parts.

1. Complementary fu^n. (C.F.)
2. Particular integral (P.I.).

$$y = \text{C.F.} + \text{P.I.}$$

C.F. ↓ $f(D)\,y = 0.$

P.I. ↓ $\dfrac{1}{f(D)}\,X.$

• n-th order DE. → C.F. gets n constants.

→ P.I. no const.

finding C.F.

$$f(D)\,y = 0.$$

$$y = e^{mx}$$

$$\boxed{f(m) = 0.}$$

$$\boxed{f(D)y=0}$$

$$\Rightarrow \left(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n\right) y = 0.$$

$$\boxed{y = e^{mx}.}$$

$$Dy = m e^{mx}.$$

$$D^2 y = m^2 e^{mx}.$$

$$D^3 y = m^3 e^{mx}.$$

$$\vdots$$

$$\underbrace{\left(m^n + P_1 m^{n-1} + P_2 m^{n-2} + \cdots + P_n\right)}\, e^{mx} = 0$$

$$f(m) = 0.$$

$$\boxed{f(m)=0.} \rightarrow m = m_1, m_2, \ldots, m_n$$

real & distinct.

$$C.F = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots + C_n e^{m_n x}$$

$\rightarrow$ (m_1, m_2 equal).

$$C.F = (C_1 + C_2 x) e^{m_1 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$$

→ ($\underline{m_1, m_2, m_3}$ equal)

$$C.F. = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_4 x} + \dots + C_n e^{m_n x}.$$

→ When auxiliary eqn.

$$f(m) = 0.$$

has imaginary roots.

$$\begin{cases} m_1 = \alpha + i\beta \\ m_2 = \alpha - i\beta . \end{cases}$$

$$C.F. = e^{\alpha x} [c_1 \cos\beta x + c_2 \sin\beta x].$$

$$C.F. = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= e^{\alpha x} [C_1 e^{i\beta x} + C_2 e^{-i\beta x}]$$

$$= e^{\alpha x} [C_1 (\cos\beta x + i \sin\beta x) + C_2 (\cos\beta x - i \sin\beta x)]$$

$$= e^{\alpha x}\left[\underbrace{(C_1+C_2)}_{A}\cos\beta x + \underbrace{i(C_1-C_2)}_{B}\sin\beta x\right]$$

$$= e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right],$$

A, B consts.

- $m_1 = m_2 = m$

why? $\boxed{C.F. = (C_1 + C_2 x)e^{mx}}$

$(D-m)^2 y = 0.$ $\leftarrow$ $f(D) = 0.$

$y = e^{lx}$

$f(l) = 0$

$(D-m)\underline{(D-m)y} = 0.$

$(D-m)v = 0$, when $v = (D-m)y$.

$\frac{dv}{dx} - mv = 0.$ $\rightarrow$ linear in v.

$$\text{I.F} = e^{-\int m\,dx} = e^{-mx}.$$

$$d\left(e^{-mx} v\right) = 0.$$

$$e^{-mx} v = c, \quad c \text{ const}.$$

$$\boxed{v = ce^{mx}.}$$

$$(D-m)y = c\,e^{mx}$$

$\downarrow$ linear in y.

$$\text{I.F.} = e^{-\int m\,dx} = e^{-mx}.$$

$$d\left(e^{-mx} y\right) = c\,e^{mx} \cdot e^{-mx}\,dx$$

$$e^{-mx} y = cx + c'$$

$$y = (cx + c')\,e^{mx}.$$

Example: $\dfrac{d^2y}{dx^2} + 8\dfrac{dy}{dx} + 25y = 0.$

$$(D^2 + 8D + 25)\,y = 0,$$

$$D \equiv \frac{d}{dx}.$$

put $y = e^{mx}$

auxiliary eqn.

$$\boxed{m^2 + 8m + 25 = 0.}$$

$$m = \frac{-8 \pm \sqrt{64 - 100}}{2}$$

$$= -4 \pm 3i$$

$$\text{C.f.} = e^{-4x}\left[A\cos 3x + B\sin 3x\right]$$

A, B arb. consts.

Finding P.I.

$f(D)\, y = X$

$$\text{P.I.} = \frac{1}{f(D)} X.$$

$X =$
- $\to e^{ax}$, a const.
- $\to x^m$, m is +ve integ
- $\to \sin ax$, $\cos ax$.
- $\to e^{ax} V$, $V = V(x)$.
- $\to$

I.) $$\text{P.I.} = \frac{1}{f(D)} e^{ax}, \quad a \text{ const.}$$

$$= \begin{cases} \dfrac{1}{f(a)} e^{ax}, \text{ if } f(a) \neq 0 \\ \dfrac{1}{\phi(a)} \dfrac{x^l}{l!} e^{ax} \text{ if } f(a) = 0 \end{cases}$$

Dy, $D \equiv \frac{d}{d}$

$\frac{1}{D} y$

$\frac{1}{D^2} y$

If $f(D) = (D-a)^l \phi(D)$,
$\phi(a) \neq 0$.

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^l \phi(D)} e^{ax}.$$

$$= e^{ax} \cdot \frac{1}{\phi(a)} \cdot \frac{1}{(D-a)^l} \cdot 1.$$

$$\frac{1}{D-a} 1. = \int 1 \, dx = x.$$

$$\frac{1}{(D-a)^2} \cdot 1 = \int x \, dx = \frac{x^2}{2}$$

$$\frac{1}{(D-a)^3} \cdot 1 = \int \frac{x^2}{2} \cdot dx = \frac{x^3}{3!}$$

Example:

$$D^3 y + y = 3 + e^{-x} + 5e^{2x}.$$

C.F. put $y = e^{mx}$

auxiliary eqn.

$$m^3 + 1 = 0$$

$$(m+1)(m^2 - m + 1) = 0$$

$$\boxed{m = -1, \ \frac{1 \pm \sqrt{3} i}{2}}$$

$$\text{C.F.} = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}x\right)\right],$$

c_1, c_2, c_3 consts.

find <u>P.I.</u>