1st. order 1st. degree ODE.

7.9.16

Example:

① — $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$$M\,dx + N\,dy = 0$$

$$M = x^2y - 2xy^2, \quad N = -(x^3 - 3x^2y).$$

$$\frac{\partial M}{\partial y} \overset{?}{=} \frac{\partial N}{\partial x}. \quad (\text{Not}).$$

① is not exact.

Rule 1 : homogeneous & $Mx + Ny \neq 0$

$\frac{1}{Mx+Ny}$ is an IF.

$$Mx + Ny = x^2y^2 \neq 0$$

$\frac{1}{x^2y^2}$ is an IF.

$$\frac{x^2y - 2xy^2}{x^2y^2}\,dx - \frac{(x^3 - 3x^2y)}{x^2y^2}\,dy = 0$$

general soln. is

$$\int \frac{x^2y - 2xy^2}{x^2y^2}\,dx + \int \frac{3}{y}\,dy = c,$$

c is a const.

$\cdot\ Mdx + Ndy = 0$ exact ODE

gen. sol^n: is

$$\int M dx + \int (\text{terms of } N \text{ free from } x)\, dy = c.$$

$$\frac{x}{y} + \log \frac{y^3}{x^2} = c$$

$y\,dx + x\,dy = d(xy)$

<u>Example:</u>

$$\underbrace{y\,(xy + 2x^2y^2)}_{M}dx + \underbrace{x\,(xy - x^2y^2)}_{N}dy = 0.$$

$f_1(xy)\, y\,dx + f_2(xy)\, x\,dy = 0$

<u>Rule II</u> if $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an IF.

$Mx - Ny = 3x^3y^3.$

$\frac{1}{3x^3y^3}$ is an IF.

$$\frac{y(xy + 2x^2y^2)}{3x^3y^3}dx + \frac{x(xy - x^2y^2)}{3x^3y^3}dy = 0.$$

is exact DE.

get the ~~~~ general sol^n.

<u>Rule III</u> $\quad M\,dx + N\,dy = 0.$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x), \text{ a fn. of only } x.$$

$$\boxed{e^{\int f(x)\,dx} = e^{\int \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N \, dx}}$$ is an IF.

<u>Rule IV</u>

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = g(y), \text{ a fn. of only } y.$$

$$\boxed{e^{\int g(y)\,dy}}$$ is an I.F.

<u>Example:</u> $\underbrace{(3x^2y^4 + 2xy)}_{M}dx + \underbrace{(2x^3y - x^2)}_{N}dy = 0.$

$$\frac{\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}}{M} = -\frac{2}{y}, \text{ a fn. of } y \text{ only.}$$

$\therefore \quad e^{-\int \frac{2}{y} dy}$ is an IF.

$= e^{-2\log y} = \frac{1}{y^2}$

$$\frac{3x^2y^4 + 2xy}{y^2} dx + \frac{2x^3y - x^2}{y^2} dy = 0.$$

is an exact DE

and the general solⁿ is given by

$$\int \left(3x^2y^2 + \frac{2x}{y}\right) dx + 0 = c, \ c \text{ const.}$$

$$\boxed{x^3y^3 + x^2 = cy}$$

- $\boxed{M\,dx + N\,dy = 0,}$ not exact

i) $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = f(x), \quad IF = e^{\int f(x)\,dx}$$

ii) $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{N} = g(y), \quad IF = e^{\int g(y)\,dy}$$

$\boxed{\text{Why?}}$

Let μ be an IF, $\boxed{\mu = \mu(x,y)}$ ~~that~~

~~make~~ for $M\,dx + N\,dy = 0$.

$\Rightarrow$ $(M\mu)\,dx + (N\mu)\,dy = 0$ is exact

i.e. $$\frac{\partial}{\partial y}(M\mu) = \frac{\partial}{\partial x}(N\mu).$$

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial x} + \mu\frac{\partial N}{\partial x}.$$

or $$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} = \mu\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

or $$M\frac{\partial}{\partial y}(\log \mu) - N\frac{\partial}{\partial x}(\log \mu) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \qquad \text{——— (1)}$$

(i) let $\mu = \underline{\underline{\mu(x)}}$

Then (1) $\Rightarrow$ $-N \frac{d}{dx}(\log \mu) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

or $\underset{\mu(x)}{\underset{\parallel}{\mu}} = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx}$.

(ii) $\mu = \mu(y) \longrightarrow$

- hom. $\longrightarrow$ $\underline{\underline{y = vx}}$
- $\hookrightarrow$ Rule 1

$Mx + Ny \neq 0$, $\frac{1}{Mx+Ny}$ is an IF.

Rule V

$$\underbrace{x^{\alpha_1} y^{\beta_1}}\underline{(m_1 y\,dx + n_1 x\,dy)}$$
$$+ \underbrace{x^{\alpha_2} y^{\beta_2}}\underline{(m_2 y\,dx + n_2 x\,dy)} = 0.$$

$$IF_1 = x^{k_1 m_1 - \alpha_1 - 1} \; y^{k_1 n_1 - \beta_1 - 1}$$

$$IF_2 = x^{k_2 m_2 - \alpha_2 - 1} \; y^{k_2 n_2 - \beta_2 - 1}.$$

Common integrating factor

find k_1, k_2 s.t.

$$\begin{cases} k_1 m_1 - \alpha_1 - 1 = k_2 m_2 - \alpha_2 - 1. \\ k_1 n_1 - \beta_1 - 1 = k_2 n_2 - \beta_2 - 1. \end{cases}$$

Determine k_1, k_2.

Example: $(y^2 + 2x^2 y)\,dx + (2x^3 - xy)\,dy = 0.$

Apply Rule V to find an IF.

Sol^n.

~~IF~~

$$y\,(y\,dx - x\,dy) + x^2\,(2y\,dx + x\,dy) = 0.$$

$$IF_1 = x^{k_1(1) - 0 - 1} \; y^{k_1(-1) - 1 - 1}$$

$$IF_2 = x^{k_2(2)-2-1}\, y^{k_2(2)-0-1}.$$

Common IF can be obtained by letting

$$\cancel{k_1} = 2k_2 - 2$$

$$\cancel{-k_1 - 1} = 2k_2$$

$$-1 = 4k_2 - 2 \Rightarrow k_2 = 1/4$$

$$\therefore k_1 = \cancel{2}\cdot\frac{1}{\cancel{4}_2} - 2 = -3/2$$

$$\therefore \text{Common } IF = x^{-\frac{3}{2}-1}\, y^{3/2-2}$$

$$= x^{-5/2}\, y^{-1/2}.$$

find the general sl^n.

$$\int (y^2 + 2x^2y)\, x^{-5/2}\, y^{-\frac{1}{2}}\, dx + 0 = c,$$

c const.

Linear DE. (1st. order)

$$\boxed{\frac{dy}{dx} + Py = Q}\ ;$$

$P = P(x)$ or const.
$Q = Q(x)$ or const.

↳ linear DE in y.

↳ linear in the dependent variable y.

$$\boxed{I.F = e^{\int P\,dx}}$$

why?

$$(Py - Q)\,dx + dy = 0.$$

$M = \overset{\to x}{P}y - \overset{\to x}{Q}$, $N = 1$.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{P - 0}{1} = P.$$

a fn. of only x.

Apply Rule III

$$e^{\int P\,dx}$$

Example:

$$\cos^2 x \frac{dy}{dx} + y = \tan x.$$

$$\frac{dy}{dx} + \underbrace{\sec^2 x}_{P}\, y = \underbrace{\tan x \sec^2 x}_{Q}.$$

linear in y.

gen. sol^n.

$$y = (\tan x - 1) + c e^{-\tan x}, \quad c \text{ const.}$$

Example: Solve

$$(1+y^2)dx = (\tan^{-1}y - x)\,dy$$

$$\frac{dy}{dx} = \frac{1+y^2}{\tan^{-1}y - x}.$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

or $$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$$

$\hookrightarrow$ linear in x.

get the gen. sol^n.