5.9.16

Lagrange's multiplier method.

max/min

$$f(x_1, \ldots, x_n, u_1, \ldots, u_m).$$

s.t. $\left.\boxed{\begin{array}{l} g_1 = 0 \\ g_2 = 0 \\ \vdots \\ g_m = 0. \end{array}}\right\}$ $g_i = g_i(x_1, \ldots, x_n, u_1, \ldots, u_m)$.

$1 \times \underline{df = 0}$. for stationary pts.

$\lambda_1 \times dg_1 = 0$

$\lambda_2 \times dg_2 = 0$

$\vdots \quad \vdots$

$\lambda_m \times dg_m = 0$

$x_1, \ldots, x_n$, $u_1, \ldots, u_m$, $\lambda_1, \ldots, \lambda_m$ $\} \rightarrow n + 2m$

. $\boxed{dL = 0}$ where $\boxed{L = f + \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m.}$

i.e. $\left.\begin{array}{l} L_{x_1} = 0,\ L_{x_2} = 0 \ldots L_{x_n} = 0 \\ L_{u_1} = 0,\ L_{u_2} = 0 \ldots L_{u_m} = 0. \\ g_1 = 0,\ g_2 = 0 \ldots g_m = 0. \end{array}\right\}$ $n + m + m$ $= n + 2m.$

. $\boxed{\begin{array}{ll} d^2 L > 0 & \rightarrow \min \\ \quad\ \ < 0 & \rightarrow \max. \end{array}}$

Example: Use Lagrange's method to find the shortest distance from the point $(0,b)$ to the parabola $x^2=4y$.

Sol^n.

min

$$f(x,y) = (x-0)^2+(y-b)^2$$

s.t. $x^2=4y.$ $\qquad$ $x^2-4y=0$

let $\quad L(x,y) = x^2+(y-b)^2+\lambda(x^2-4y).$

$$\begin{cases} L_x=0 \\ L_y=0 \end{cases}$$

$L_x = 2x+2\lambda x = 0 \Rightarrow x=0$ or $\lambda=-1$

$L_y = 2(y-b)-4\lambda=0 \Rightarrow y=b+2\lambda.$

$x^2-4y=0$

Case 1 $\quad x=0, y=0.$ $\quad (0,0).$

$\lambda=-b/2$

$$d^2L = L_{xx}(dx)^2+L_{yy}(dy)^2+2L_{xy}\,dx\,dy$$

$$=(2+2\lambda)(dx)^2+2(dy)^2+0$$

>0 if $2+2.(-\frac{b}{2})>0$

i.e. $\boxed{b<2}$

min at $(0,0)$ provided $b<2$

Case 2 $\quad \lambda=-1$

$y=b-2$

$x=\pm2\sqrt{b-2}$

$(\pm2\sqrt{b-2},\ b-2),\ b\geq2.$

min or max $\rightarrow$ $d^2L>0$??

• if $b < 2$, $(1,0)$ is the only critical pt.
& min & S.D $= b$

• if $b \geq 2$, $(\pm 2\sqrt{b-2},\ b-2)$ are the critical pts, min., $\underline{S.D = 2\sqrt{b-1}.}$

Example: Maximize $\underline{x^a y^b z^c}$, where a, b, c are +ve constants s.t. the condition

$$x^k + y^k + z^k = 1$$

where x, y, z are non-negative variables & $k > 0$.

Sol^n: Max $x^a y^b z^c$

max $a\log x + b\log y + c\log z$

s.t. $\boxed{x^k + y^k + z^k = 1.}$

let $L = a\log x + b\log y + c\log z - \lambda(x^k + y^k + z^k - 1)$

$$L_x = 0 = \frac{a}{x} - \lambda k x^{k-1} \implies \lambda x^k = \frac{a}{k}$$
$$L_y = 0 = \frac{b}{y} - \lambda k y^{k-1} \implies \lambda y^k = \frac{b}{k}$$
$$L_z = 0 = \frac{c}{z} - \lambda k z^{k-1} \implies \lambda z^k = \frac{c}{k}$$

$$\lambda(x^k + y^k + z^k) = \frac{a+b+c}{k}.$$

$$\Rightarrow \lambda = \frac{a+b+c}{k} \quad \text{as } x^k+y^k+z^k = 1$$

$$\therefore \boxed{x^k = \frac{a}{a+b+c}, \quad y^k = \frac{b}{a+b+c}, \quad z^k = \frac{c}{a+b+c}}$$

gives the stationary pts. (x, y, z)

To check max/min $\quad d^2 L$

$$L_{xx} = -\frac{a}{x^2} - \lambda k(k-1)x^{k-2}$$

$$= -\frac{a - \lambda k(k-1)x^k}{x^2}$$

$$= \frac{-\cancel{a} - (k-\cancel{1})a}{x^2}$$

$$= -\frac{ka}{x^2}$$

$$L_{yy} = -\frac{kb}{y^2}$$

$$L_{zz} = -\frac{kc}{z^2}$$

$$L_{xy} = 0 = L_{yz} = L_{zx}$$

$$\left. d^2 L \right]_{\text{at critical pts.}} = L_{xx}(dx)^2 + L_{yy}(dy)^2 + L_{zz}(dz)^2 + 2L_{xy}\,dx\,dy + 2L_{xz}\,dx\,dz + 2L_{yz}\,dy\,dz.$$

$$= -\frac{ak}{x^2}(dx)^2 - \frac{bk}{y^2}(dy)^2 - \frac{ck}{z^2}(dz)^2.$$

$$x^k + y^k + z^k = 1.$$

$$\Rightarrow k x^{k-1} dx + k y^{k-1} dy + k z^{k-1} dz = 0.$$

$$\Rightarrow dz = -\frac{x^{k-1}dx + y^{k-1}dy}{z^{k-1}}.$$

$$= -\frac{ak}{x^2}(dx)^2 - \frac{bk}{y^2}(dy)^2 - \frac{ck}{z^2}\left(\frac{x^{k-1}dx + y^{k-1}dy}{z^{k-1}}\right)^2$$

< 0 as $a, k, b, c > 0$.

So max. value

$$\boxed{x^a y^b z^c \le \sqrt[k]{\left(\frac{a}{a+b+c}\right)^a \left(\frac{b}{a+b+c}\right)^b \left(\frac{c}{a+b+c}\right)^c}.}$$

(ii) Hence prove that for any six +ve real numbers u, v, w, a, b, c,

$$\left(\frac{u}{a}\right)^a \left(\frac{v}{b}\right)^b \left(\frac{w}{c}\right)^c \le \left(\frac{u+v+w}{a+b+c}\right)^{a+b+c}.$$

(GM $\le$ A.M.)

$$x^k = \frac{u}{u+v+w}, \quad y^k = \frac{v}{u+v+w},$$

$$z^k = \frac{w}{u+v+w}$$

(4)

$$\frac{u^a \cdot v^b \cdot w^c}{(u+v+w)^{a+b+c}} \leq \frac{a^a \, b^b \, c^c}{(a+b+c)^{a+b+c}}.$$

$$\left(\frac{u}{a}\right)^a \left(\frac{v}{b}\right)^b \left(\frac{w}{c}\right)^c \leq \left(\frac{u+v+w}{a+b+c}\right)^{a+b+c}.$$

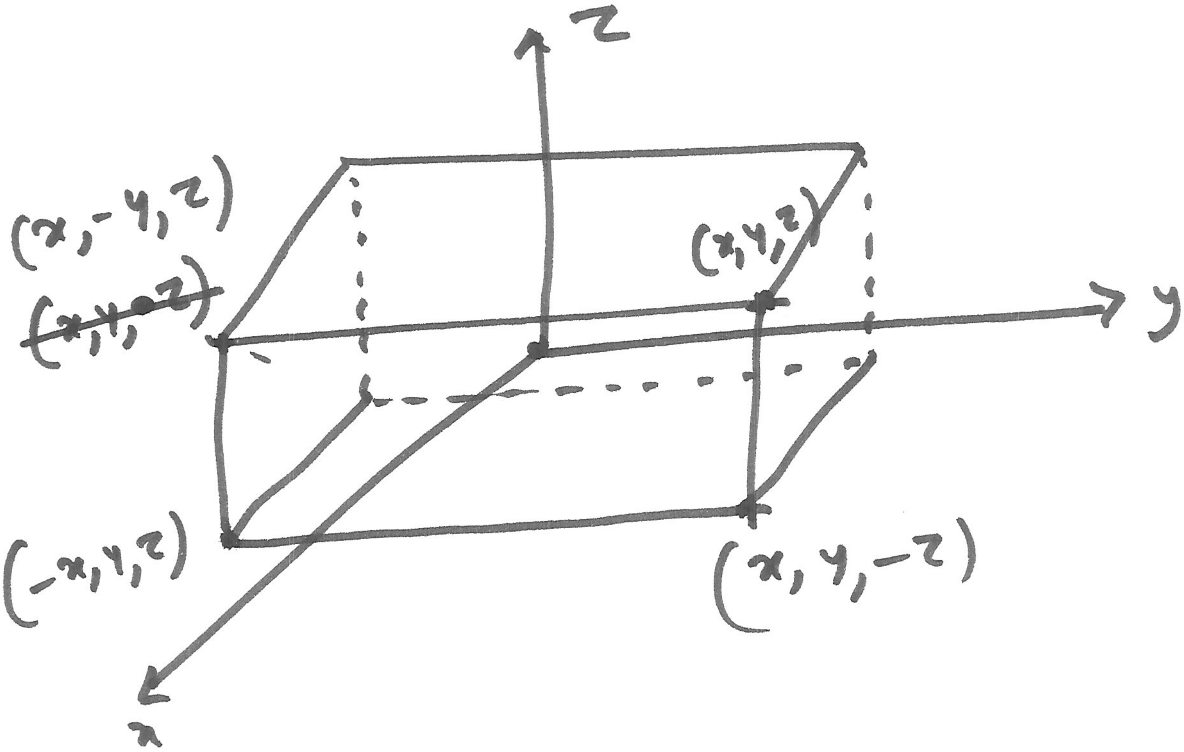
Proved.

Example: Prove that the volume of the greatest rectangular parallelopied that can be inscribed in the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{is} \quad \frac{8abc}{3\sqrt{3}}.$$

max. $V = (2x)(2y)(2z)$

s.t. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$



- $dL = 0$
- $d^2L \geq 0$.

Ordinary Differential equation

(only one independent variable).

Example:

$$(y+1)^2 \frac{dx}{dz} + z\frac{dy}{dz} - (y+a) = 0.$$

→ z is the only independent variable.

x, y dependent variables.

all differential co-efficients → single ind. variable.

$\frac{d}{dz}$ →

Partial differential eqn. (more than one ind. variabl

Example:

$$y^2 \frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = nxz$$

→ x, y are ind. variables

z is dep. variable.

- order of a DE ?
- degree of a DE ?

Example:

(i) $$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = r.$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

order = 2
degree = 2.

(ii) $$y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}$$

order = 1
degree = 2

Example: ~~Form~~ From $x^2+y^2+2ax+2by+c=0$, derive a DE not containing a, b or c.

Soln. $$x^2+y^2+2ax+2by+c=0 \quad \text{--- (1)}.$$

w.r.to x $$2x+2y\frac{dy}{dx}+2a+2b\frac{dy}{dx}=0 \checkmark$$

w.r.to x $$1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}+b\frac{d^2y}{dx^2}=0. \checkmark$$

w.r.to x $$2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}\frac{d^2y}{dx^2}+y\frac{d^3y}{dx^3}+b\frac{d^3y}{dx^3}=0.$$

$$\frac{d^3y}{dx^3}\left[1+\left(\frac{dy}{dx}\right)^2\right] - 3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2 = 0. \quad \text{--- (2)}.$$

(eliminating b).

Complete integral / general Soln.

- (1) is called the complete integral of the DE (2).

- $\boxed{\text{\# of consts. in a complete integral of a DE = order of the DE.}}$

Particular Solns. → giving particular values to the consts. in the complete int. of the DE.

Exercise: Form the DE of which

$e^{2y} + 2cx\,e^{y} + c^2 = 0$ is the complete integral.

order = ?

degree = ?

DE of 1st. order, 1st. degree

1. Equn! of the form $f_1(x)\,dx + f_2(y)\,dy = 0$.

(variable are seperable).

Example: $(1-x)\,dy - (1+y)\,dx = 0$.

2. Equn. homogeneous in x, y.

$$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}\,, \quad f_1, f_2 \text{ homogeneous in } x, y \text{ of same degree.}$$

$\downarrow$ put

$$y = vx.$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Example: $(x^2+y^2)\,dx - 2xy\,dy = 0$.

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}.$$

Let $\boxed{y = vx}$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{1+v^2}{2v}.$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1-v^2}{2v}.$$

$$\therefore\ \frac{2v}{1-v^2}\,dv = \frac{dx}{x}$$

Sol^n: $x^2-y^2=cx$, c arb. const.

3. Non-homogeneous equ^{n}s of first degree in x, y

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$$

put $x = x'+h$
$y = y'+k$.

$$\frac{dy}{dx} = \frac{dy'}{dx'}.$$

$$\frac{dy'}{dx'} = \frac{ax'+by'+c+ah+bk}{a_1x'+b_1y'+c_1+a_1h+b_1k}$$

Choose h, k so that $\boxed{\begin{aligned} ah+bk+c&=0\\ a_1h+b_1k+c_1&=0\end{aligned}}$ & find h, k.

Then we get

$$\frac{dy'}{dx'} = \frac{ax'+by'}{a_1x'+b_1y'}$$

put $y' = vx'$ & proceed.

Example: Solve $(3y-7x+7)\,dx + (7y-3x+3)\,dy = 0$.

$$\frac{dy_1}{dx_1} = -\frac{3y_1+7x_1}{7y_1-3x_1}$$

~~$x_1 = \cdot$~~

$x = x_1 + h$

$y = y_1 + k$

$h = ?$ $\quad 3k - 7h + 7 = 0$

$k = ?$ $\quad 7k - 3h + 3 = 0.$

4. Exact DE.

$M\,dx + N\,dy = 0$, M, N fun's of x, y.

is exact iff $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$.

$\boxed{M\,dx + N\,dy}$ is exact if $\exists\ u = u(x,y)$

s.t. $M\,dx + N\,dy = du = u_x\,dx + u_y\,dy$

$$\Rightarrow \quad u_x = M \qquad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$$
$$u_y = N \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Soln.

$\int M\,dx + \int$ (terms of N free from x) $dy = c$,

c is a const.

Example:

$$\underbrace{(x^2 - 4xy - 2y^2)}_{M}dx + \underbrace{(y^2 - 4xy - 2x^2)}_{N}dy = 0. \quad \text{——(1)}$$

$$\frac{\partial M}{\partial y} = -4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x = \frac{\partial M}{\partial y}.$$

So (1) is exact DE.

$\therefore$ Complete integral is given by

$$\int (x^2 - 4xy - 2y^2)dx + \int y^2\,dy = c, \; c \text{ const.}$$

i.e. $\frac{x^3}{3} - \frac{4x^2}{2}y - 2y^2x + \frac{y^3}{3} = c.$

Integrating ~~factor~~ factor (IF).

__Example:__ $\boxed{y\,dx - x\,dy = 0} \rightarrow$ not exact.

$M = y, \quad N = -x.$

- but when multiplied by $\frac{1}{y^2}$

$$\frac{y\,dx - x\,dy}{y^2} = 0.$$

i.e. $d\left(\frac{x}{y}\right) = 0. \rightarrow \frac{x}{y} = c$, const.

- when multiplied by $\frac{1}{xy}$.

$$\frac{dx}{x} - \frac{dy}{y} = 0. \rightarrow \text{exact.}$$

$$M = \frac{1}{x}, \ N = -\frac{1}{y}$$

$$\log x - \log y = \log c.$$

- $\frac{1}{x^2}$ is also an IF.
- # of IFs infinite.

$M\,dx + N\,dy = 0.$

~~exact~~ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial y}$

$\downarrow$

exact.

$\downarrow$

complete integral is

$$\int M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = c.$$

$\boxed{\text{why?}}$

Exact DE. (1st. order 1st. degree).

$$M\,dx + N\,dy = 0 = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

for some f^n. $u = u(x,y)$

$$\boxed{M = \frac{\partial u}{\partial x}}, \quad N = \frac{\partial u}{\partial y}, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

as exact.

$\Downarrow$

$$u(x,y) = \int M\,dx + \phi(y) \quad \text{——} \quad (1).$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int M\,dx + \phi'(y).$$

$$N = \int \frac{\partial M}{\partial y}\,dx + \phi'(y)$$

$$= \int \frac{\partial N}{\partial x}\,dx + \phi'(y).$$

- $M, N \to$ continuous diff. fun^{ns}.
- $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x} \to$ continuous in ~~same~~ region.

$$\Rightarrow \phi'(y) = N - \int \frac{\partial N}{\partial x}\,dx.$$

Hence

$$u(x,y) = \int M\,dx + \phi(y) \quad (\text{by } ①)$$

$$c = \int M\,dx + \int \left[N - \left(\int \frac{\partial N}{\partial x}\,dx\right)\right] dy$$

term of N containing x.

. $du = 0 \Rightarrow u = c$ in the sol^n.

Examples

$$N = \frac{y^2 - 3x^2}{y^4}.$$

$$\frac{\partial N}{\partial x} = -\frac{6x}{y^4}$$

$$\int \frac{\partial N}{\partial x} = -\frac{6x^2}{2y^4} = -\frac{3x^2}{y^4}$$

Rules for finding IFs.

$$\boxed{M\,dx + N\,dy = 0}$$

- **By inspection**

Example:

$$y\,dx - x\,dy + \log x\,dx = 0 \quad \text{--- (1)}$$

$$M = y + \log x, \qquad N = -x$$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = -1$$

(1) Not exact.

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x}{x^2}\,dx = 0.$$

$$-d\left(\frac{y}{x}\right) + \frac{\log x}{x^2}\,dx = 0.$$

$$cx + y + \log x + 1 = 0 \quad \text{(check)}.$$

Example: $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0.$

$$\frac{y\,dx + x\,dy}{x^2y^2} + \frac{xy^2\,dx - x^2y\,dy}{x^2y^2} = 0.$$

$$\frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0.$$

Rule I. $\quad M\,dx + N\,dy = 0.$

if $Mx + Ny \neq 0$, eqn. is homogeneous

then $\dfrac{1}{Mx+Ny}$ is an IF.

Rule II $\quad \underbrace{f_1(xy)\,y}_{M}\,dx + \underbrace{f_2(xy)\,x}_{N}\,dy = 0.$

if $Mx - Ny \neq 0$, then $\dfrac{1}{Mx-Ny}$ is an IF.

Rule III $\quad M\,dx + N\,dy = 0.$

if $$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{\cancel{M}\ N} = f(x),$$ a fn. of x alone.

then $e^{\int f(x)\,dx}$ is an IF.

Rule IV

if
$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y), \text{ a fn. of } y \text{ alone}$$

then $e^{\int f(y)\,dy}$ is an IF.

Example: $(x^2 + y^2)\,dx - 2xy\,dy = 0.$ — (1)

$M = x^2 + y^2, \quad N = -2xy.$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y + 2y}{-2xy} = -\frac{2}{x}.$$

$$\therefore IF = e^{-\int \frac{2}{x}\,dx} = e^{-2\log x} = \frac{1}{x^2}.$$

$\frac{1}{x^2}$ makes it exact

find the soln.