31.8.16

Constrained Extremes

(for fu^{ns}. of several variables s.t. certain constraints)

Example: Find the shortest distance from the point (a, b, c) to the plane

$$px + qy + rz - s = 0.$$

Sol^n. min

$$(x-a)^2 + (y-b)^2 + (z-c)^2$$

s.t. $\boxed{px + qy + rz - s = 0.}$

let $\phi(x, y) = (x-a)^2 + (y-b)^2 + (z-c)^2$

when $px + qy + rz - s = 0.$

$$p\,dx + q\,dy + r\,dz = 0.$$

$$\Rightarrow dz = -\frac{p}{r}dx - \frac{q}{r}dy$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{p}{r}, \quad \frac{\partial z}{\partial y} = -\frac{q}{r}.$$

$$\phi_x = 2(x-a) + 2(z-c)\cdot\frac{\partial z}{\partial x}$$

$$= 2(x-a) + 2(z-c)\left(-\frac{p}{r}\right) = 0.$$

$$\phi_y = 2(y-b) + 2(z-c)\frac{\partial z}{\partial y}$$

$$= 2(y-b) + 2(z-c)\left(-\frac{q}{r}\right) = 0.$$

For stationary pts. / critical pts, we have

$$\left.\begin{array}{l}\phi_x = 0\\ \phi_y = 0\end{array}\right\} \Rightarrow \frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}.$$

$$= \frac{p(x-a)+q(y-b)+r(z-c)}{p^2+q^2+r^2}$$

$$= \frac{s-pa-qb-zc}{p^2+q^2+r^2}$$

$$= R \text{ (say)}.$$

$$\therefore \boxed{x = a+pR,\quad y = b+qR,\quad z = c+rR}$$

critical pt.

$$\phi_{xx} = 2 - 2\frac{p}{r}\cdot\frac{\partial z}{\partial x} = 2 + 2\frac{p^2}{r^2}$$

$$\phi_{yy} = 2 + 2\frac{q^2}{r^2}, \quad \phi_{xy} = -\frac{2q}{r}\frac{\partial z}{\partial x}$$

$$= 2pq/r^2$$

$$\begin{vmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{vmatrix} = 4\left(1+\frac{p^2}{r^2}\right)\left(1+\frac{q^2}{r^2}\right) - \frac{4p^2q^2}{r^4}.$$

$$= 4\left(1+\frac{p^2}{r^2}+\frac{q^2}{r^2}\right) > 0.$$

$\Rightarrow$ ϕ has min. at

$x = a+pR, \quad y = b+qR, \quad z = c+rR$

$$\phi_{min} = (pR)^2 + (qR)^2 + (rR)^2$$

$$= (p^2+q^2+r^2)\cdot\left\{\frac{s-pa-qb-rc}{p^2+q^2+r^2}\right\}^2$$

$$= \frac{(s-pa-qb-rc)^2}{p^2+q^2+r^2}.$$

General Rule.

To find stationary pts. of

$f(x_1, \dots, x_n; u_1, \dots, u_m) \rightarrow$ $n+m$ variables

s.t.
$$\boxed{\left.\begin{array}{l} g_1 = 0 \\ g_2 = 0 \\ \vdots \\ g_m = 0. \end{array}\right\}} \text{ — } (1) \rightarrow m \text{ constraints.}$$

when $g_i = g_i(x_1, \dots, x_n; u_1, \dots, u_m)$.

- Solve $u_1, u_2, \dots, u_m$ in terms of $x_1, x_2, \dots, x_n$ from eqn. (1)

$$\left\{\begin{array}{l} u_1 = \phi_1(x_1, \dots, x_n) \\ u_2 = \phi_2(x_1, \dots, x_n) \\ \vdots \\ u_m = \phi_m(x_1, \dots, x_n). \end{array}\right. \quad \Bigg| \quad J = \frac{\partial(g_1, \dots, g_m)}{\partial(u_1, \dots, u_m)} \neq 0.$$

- Then f becomes a fn. of $x_1, \dots, x_n$
- $df = 0$ at stationary pts.
- $dg_i = 0$, $i = 1, 2, \dots, m$.

$\shortparallel$

$$\Downarrow$$

$$\frac{\partial g_i}{\partial x_1} dx_1 + \frac{\partial g_i}{\partial x_2} dx_2 + \cdots + \frac{\partial g_i}{\partial x_i} dx_i$$

$$+ \frac{\partial g_i}{\partial u_1} du_1 + \cdots + \frac{\partial g_i}{\partial u_m} du_m = 0,$$

$$i = 1, 2, \ldots, m.$$

$$\Downarrow$$

find $du_1, \ldots, du_m$ in terms of $dx_1, \ldots, dx_n$

& substitute in $df = 0$.

$$0 = df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$$

$$+ \frac{\partial f}{\partial u_1} du_1 + \cdots + \frac{\partial f}{\partial u_m} du_m.$$

$$\Downarrow$$

equate co-efficients of $dx_i = 0$, $i = 1, 2, \ldots, n$

n eqns.

$f \rightarrow$ $n+m$ variables.

$g_i = 0 \rightarrow$ m constraints

$\hookrightarrow$ m eqns.

<u>Example:</u> A <u>rectangular box</u>, open at the top, is to have a volume of 32 cu ft. What must be the dimensions so that the total surface is a <u>minimum</u>.

<u>Sol^n.</u> $x \to$ length, $y \to$ breadth, $z \to$ height.

min $\quad S = xy + 2yz + 2xz$.

s.t. $\quad V = xyz = 32$.

$$S(x,y) = xy + 2(y+x) \cdot \frac{V}{xy}.$$

$\left.\begin{matrix} S_x = 0 \\ S_y = 0 \end{matrix}\right\}$ $(4,4)$ is a stationary pt.

$$\begin{vmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{vmatrix}_{(4,4)} = 3 > 0$$

$\therefore\ x = 4,\ y = 4$ & so $z = \frac{V}{xy} = \frac{32}{16} = 2$ &

$S_{min} = ?$

Lagrange's Method of Undetermined Multipliers

- min/max $f(x_1, \dots, x_n, u_1, \dots, u_m)$

  s.t. $\boxed{g_1 = 0,\ g_2 = 0, \dots, g_m = 0.}$

  when $g_i = g_i(x_1, \dots, x_n, u_1, \dots, u_m)$,

  $i = 1, 2, \dots, m$

$n+2m$

$x_1, \dots, x_n$
$u_1, \dots, u_m$

- $\boxed{L} = \boxed{f} + \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_m g_m$,

  $\lambda_1, \lambda_2, \dots, \lambda_m$ are certain constants.

- $dL = 0 \longrightarrow L_{x_1} = 0 = L_{x_2} = \dots = L_{x_n} = L_{u_1} = \dots = L_{u_m}$
- $d^2L > 0 \rightarrow$ min

  $< 0 \rightarrow$ max.

$dL \neq L_{x_1} dx + L_{x_2} dx + \dots +$

Example: find the max^m of

$f(x,y,z) = x^2y^2z^2$ subject to the condition $\boxed{x^2+y^2+z^2=a^2}$, $a \neq 0$.

(x, y, z are +ve).

Using Lagrange's multiplier method.

Sol^n: Construct a Lagrangian fn^n.

$$L(x,y,z) = x^2y^2z^2 + \lambda(x^2+y^2+z^2-a^2),$$

λ is the Lagrangian undetermined multiplier.

The stationary pts. are given by $dL=0$

$$\left.\begin{aligned} L_x &= 2xy^2z^2 + 2x\lambda = 0 \\ L_y &= 2yx^2z^2 + 2y\lambda = 0 \\ L_z &= 2zx^2y^2 + 2\lambda z = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x(y^2z^2+\lambda)&=0 \\ y(x^2z^2+\lambda)&=0 \\ z(x^2y^2+\lambda)&=0 \end{aligned}$$

$x=0, y=0, z=0 \rightarrow$ ignore as $x^2+y^2+z^2 = a^2 \neq 0$

$$\lambda = \underbrace{-y^2z^2 = -x^2z^2} = -x^2y^2$$

$$\Rightarrow x^2 = y^2 = z^2 = \frac{x^2+y^2+z^2}{3} = \frac{a^2}{3}.$$

$$\Rightarrow \boxed{x = \pm \frac{a}{\sqrt{3}} = y = z}, \quad \lambda = -x^4 = -\frac{a^4}{9}.$$

$L_{xx} = 2y^2z^2 + 2\lambda = 0.$

$L_{yy} = 2x^2z^2 + 2\lambda = 0$

$L_{zz} = 2x^2y^2 + 2\lambda = 0.$

$L_{xy} = 4xyz^2 = \frac{4a^4}{9}$

$L_{xz} = 4xy^2z = \frac{4a^4}{9}$

$L_{xy} = \frac{4a^4}{9}.$

$$d^2L < 0.$$

$$\begin{aligned} d^2L &= \underset{0}{L_{xx}}(dx)^2 + \underset{0}{L_{yy}}(dy)^2 + \underset{0}{L_{zz}}(dz)^2 \\ &\quad + 2L_{xy}\,dx\,dy + 2L_{xz}\,dx\,dy \\ &\quad + 2L_{yz}\,dy\,dz. \end{aligned}$$

$$= 2 \cdot \frac{4a^4}{9}\left[\underline{\underline{dz}}\,(dx+dy) + dx\,dy\right].$$

$$= \frac{8a^4}{9}\left[-(dx+dy)^2 + dx\,dy\right].$$

$x^2+y^2+z^2 = a^2$

$2x\,dx + 2y\,dy + 2z\,dz = 0$

$\boxed{dz} = -\frac{x\,dx - y\,dy}{z}$

$< 0.$

$\cancel{\left(\pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}, \pm\frac{a}{\sqrt{3}}\right)}$

Ans. $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

or $\left(-\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}\right)$, acc. as $a > 0$ or $a < 0$.