29.8.16

Taylor's expansion of fun^{ns}. of two variables

MVTh. $f(x,y)$, f_x, f_y continuous in some nbd. (a,b).

$$f(a+h,\, b+k) - f(a,b) = h\, f_x(a+\theta h,\, b+\theta k) + k\, f_y(a+\theta h,\, b+\theta k),$$
$$(0<\theta<1).$$

Taylor's Theorem $f(x,y)$ defined over some domain D having continuous partial derivatives of order n. (or f is differentiable) in some nbd. of (a,b).

$$f(a+h,\, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \cdots$$
$$\cdots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n$$

where R_n = the remainder after n terms (due to Lagrange)

$$= \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\,(a+\theta h,\, b+\theta k),$$

$(0 < \theta < 1)$

<u>Note:</u>

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^m f\,(a,b)$$

$$= h^m \frac{\partial^m f(a,b)}{\partial x^m} + \binom{m}{1} h^{m-1}k \frac{\partial^m f(a,b)}{\partial x^{m-1}\partial y}$$

$$+ \binom{m}{2} h^{m-2} k^2 \frac{\partial^m f(a,b)}{\partial x^{m-2}\partial y^2} + \cdots$$

$$+ \binom{m}{m} k^m \frac{\partial^m f(a,b)}{\partial y^m}.$$

- put $a=0$, $b=0$, $h=x$, $k=y$ $\Rightarrow$ Maclaurin's Theorem
- put $a+h=x$, $b+k=y$ so that $h=x-a$, $k=y-b$

$\Rightarrow$ Taylor's expansion of $f(x,y)$ about the pt. (a,b)

$\Rightarrow$ Taylor's expansion of $f(x,y)$ in powers of

$x-a, \quad y-b.$

Example: Expand $f(x,y) = x^2y + 3y - 2$ about the point $(1,-2)$.

or

equivalently expand $f(x,y)$ in powers of $x-1$ and $y+2$.

Soln: Taylor's expansion of $f(x,y)$ about (a,b)

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b)$$

$$+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b)$$

$$+ \cdots + \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b)$$

$$+ \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+\theta(x-a), y+\theta(y-b))$$

$f(x,y) = x^2y + 3y - 2, \quad a = 1, \; b = -2$

$f(1,-2) = -10$

$f_x(1,-2) = -4$

$f_y(1,-2) = 4$

$f_{xx}(1,-2) = -4$

$\underline{f_{xy}}(1,-2) = 2$

$f_{yy}(1,-2) = 0$

$f_{xxx}(1,-2) = 0$

$\underline{f_{yxx}}(1,-2) = 2$

$f_{yyy}(1,-2) = 0$

$\times\, f_{xxy}(1,-2) = 2$

$f_{yxy}(1,-2) = 0$

$\times\, f_{xyy}(1,-2) = 0$

$$f(x,y) = -10 + \left[(x-1)(-4) + (y+2)4\right]$$

$$+ \frac{1}{2!}\left[(x-1)^2(-4) + (y+2)^2 0 + 2(x-1)(y+2)2\right]$$

$$+ \frac{1}{3!}\left\{(x-1)^3 0 + 3(x-1)^2(y+2)\cdot 2 + 3(x-1)(y+2)^2\cdot 0 + (x-1)^3(\cancel{y}+y)^3 \times 0\right\}$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2).$$

Example: Using Taylor's Theorem, prove

that $\displaystyle \lim_{\substack{(x,y)\to(0,0) \\ (y=-x)}} \frac{\sin xy + xe^{x} - y}{x\cos y + \sin 2y} = -2$

Soln. Let $f(x,y) = \sin xy + xe^{x} - y$.

~~$= f(0,0) + \left[x\frac{\partial}{\partial x} + y\right.$~~

$$= f(0,0) + \left[x f_x(0,0) + y f_y(0,0)\right]$$
$$+ \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right]$$
$$+ \cdots$$

$$= x - y + \frac{1}{2}\left[2x^2 + 2xy\right] + \cdots$$
(check)

Let $\phi(x,y) = x\cos y + \sin 2y$

$$= \phi(0,0) + \left[x\phi_x(0,0) + y\phi_y(0,0)\right]$$
$$+ \frac{1}{2!}\left[x^2\phi_{xx}(0,0) + 2xy\phi_{xy}(0,0) + y^2\phi_{yy}(0,0)\right] + \cdots$$

$$= x + 2y + \cdots\cdots \text{ (check)}$$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\phi(x,y)}$$

$$y = -x$$

$$= \lim_{\substack{y=-x \\ (x,y)\to(0,0)}} \frac{x - y + x^2 + xy + \cdots}{x + 2y + \cdots}$$

$$= \lim_{x\to 0} \frac{2x + \cancel{x^2} - \cancel{x^2} + \text{higher powers of } x}{x - 2x + \text{higher powers of } x}$$

$$= -2$$

- $R_n \rightarrow$ error term.

  estimate errors.

Extreme values for fun^{ns}. of Two or more variables

- $f(x,y) \Rightarrow$ real valued fu^{n}. of two ind. variables x, y defined in a certain nbd. of (a,b) of its domain.

- $f(x,y) < \underline{\underline{f(a,b)}} \quad \forall \ (x,y) \in N$ (except (a,b)), N being a suitable nbd. of (a,b)

  $\rightarrow$ f has max^{m}. at (a,b).

- $f(x,y) > f(a,b) \quad \forall \ (x,y) \in N$ (except (a,b))

  $\rightarrow$ f has min^{m}. at (a,b).

Necessary condition

- $f(x,y)$ has an extreme value at (a,b)

$$\Rightarrow f_x(a,b) = 0$$
$$f_y(a,b) = 0.$$

provided they exist.

- $f(x,y)$
- $\left.\begin{array}{l} f_x(a,b)=0 \\ f_y(a,b)=0 \end{array}\right\} \rightarrow (a,b)$ is a stationary pt.

  / critical pt.

  extreme pt. saddle pt.

Example: $f(x,y) = \begin{cases} 0 & \text{if } \underline{x=0} \text{ or } \underline{y=0} \\ 1 & \text{elsewhere.} \end{cases}$

$\cancel{f(x,y)_x =}$ $\quad f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{0-0}{h}$

$= 0.$

$\boxed{\begin{array}{l} f_x(0,0)=0 \\ f_y(0,0)=0 \end{array}}$

$f_y(0,0) = 0.$

Necessary conditions for extreme pt. is satisfied at (1,0).

But $\underline{\text{(0,0) is not an extreme pt.,}}$ it is a saddle pt. $\left[f(x,y)-f(0,0) = \begin{cases} 0 & \text{along } x \text{ or } y \text{ axis} \\ 1 & \text{elsewhere} \end{cases} \right.$

· $f(x) = |x|$.

$x=0$ <u>critical pt. or not?</u>

$f'(0)$ ~~≠~~ does not exist, still f has min. at $x=0$

<u>Saddle pt.</u> → Stationary pt. which is not an extreme pt.

↓

(a,b)

↓

$f_x(a,b)=0$, $f_y(a,b)=0$, but $f(x,y)$ has no extrema at (a,b).

<u>Note.</u> $f(x,y)$ may have extrema at (a,b) even if f_x, f_y do not exist at (a,b)

<u>Example:</u> $f(x,y) = |x| + |y|$.

↓

has ~~extrema~~ minimum value at $(0,0)$, but doesn't f_x, f_y exist at $(0,0)$.

Sufficient condition

- $f(x,y)$
- $f_x, f_y, f_{xx}, f_{yy}, f_{xy} \rightarrow$ all continuous in some nbd. of (a,b).
- $df = 0 \rightarrow \boxed{f_x = 0, \; f_y = 0}$
- $f_x(a,b) = 0, \; f_y(a,b) = 0 \rightarrow (a,b)$ is a stationary pt.
- $d^2 f = f_{xx}(dx)^2 + 2 f_{xy}\, dx\, dy + f_{yy}(dy)^2$
- $A = f_{xx}(a,b), \; B = f_{xy}(a,b), \; C = f_{yy}(a,b)$
- $H = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$.

i) $H < 0 \rightarrow$ no extrema at (a,b) (saddle pt.)

ii) $H > 0 \rightarrow f$ has extrema at (a,b)

$AC - B^2 > 0$ $\Downarrow$ A, C having same sign.

f has maxima if $A, C < 0$ or $\boxed{A < 0.}$

min. if $A, C > 0$ or $\boxed{A > 0}$

iii) $\underline{H = 0} \rightarrow$ undecided

$\rightarrow$ further investigation is needed.

$\rightarrow$ principal minors $\rightarrow A, \begin{vmatrix} A & B \\ B & C \end{vmatrix}$

$\rightarrow$ $\boxed{\text{all +ve} \rightarrow \text{min}^m}$

$\rightarrow$ $\boxed{\text{alternatively } -\text{ve}, +\text{ve} \rightarrow \text{max}^m}$

Generalization

- $f(x,y,z)$.
- $df = f_x\,dx + f_y\,dy + f_z\,dz$
- $f_x = 0,\ f_y = 0,\ f_z = 0$ at (a,b,c)

  $\rightarrow (a,b,c)$ is a stationary pt. / critical pt

  (Necessary condition).
- $d^2 f = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2 f_{xy}\,dx\,dy + 2 f_{xz}\,dx\,dz + 2 f_{yz}\,dy\,dz.$

• (Sufficient condition)

$$H = \begin{pmatrix} f_{xx} & \underline{f_{xy}} & f_{xz} \\ \underline{f_{yx}} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix}$$

<u>principal minors of H</u>

$$f_{xx}, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix}$$

alternatively $-$ve & $+$ve $\longrightarrow$ max

all $+$ve $\longrightarrow$ min

<u>Example</u>: Find the extreme values, if any, of the fn. $f(x,y) = 2(x-y)^2 - x^4 - y^4$.

<u>Soln.</u>

$$f_x = 4\cancel{(x-y)} - 4x^3 = 0 \quad \text{--- } ①.$$

$$f_y = -4\cancel{(x-y)} - 4y^3 = 0$$

$$(x+y)\cancel{(x^2 - xy + y^2)} = 0.$$

$$\begin{cases} x+y=0 \\ x-y+x^3=0. \end{cases}$$
$$2x-x^3=0.$$
$$x=0,\ \pm\sqrt{2}$$

or

$$\begin{cases} x^2-xy+y^2=0. \\ x-y-x^3=0 \end{cases}$$

No sol^n.
(why?).

$(0,0)$, $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$

(i) $(0,0)$.

$$A = f_{xx}(0,0) = 4-12x^2\Big]_{(0,0)} = 4$$
$$B = f_{xy}(0,0) = -4$$
$$C = f_{yy}(0,0) = 4-12y^2\Big]_{(0,0)} = 4.$$

$$H = AC-B^2 = 16-16=0.$$

undecided case, further investigation needed.

(ii) $(\sqrt{2}, -\sqrt{2})$

$$A = 4-24 = -20$$
$$B = -4$$
$$C = 4-24 = -20$$
$$H = AC-B^2 = 400-16 > 0.$$
$$A<0 \rightarrow \text{~~min.~~ max.}$$

$$\boxed{f_{max} = 8}$$

iii) $(-\sqrt{2}, \sqrt{2})$

$A = -20, \; B = -4, \; C = -20$

$H > 0, \quad A < C \Rightarrow$ max.

$$\boxed{f_{max} = 8}$$

Case $(0,0)$

$f(x,y) = 2(x-y)^2 - x^4 - y^4.$

along x-axis, $f(x,y) = 2x^2 - x^4$
> 0 in the nbd. of $(0,0)$.

along the line $x = y$, $f(x,y) = -2x^4 < 0$
in the nbd. of $(0,0)$

$f(x,y) - f(0,0)$ > 0
< 0.

$\Rightarrow \exists$ nbd. of $(0,0)$ when $f(x,y) - f(0,0)$ does not keep a constant sign.

$\Rightarrow f$ has no extreme value at $(0,0)$

Example: Show that $f(x,y) = x^2 - 2xy + y^2 + x^3 - y^3 + x^5$.

has a stationary pt. at $(0,0)$, but $f(0,0)$ is neither a maximum value nor a minimum value.

Sol^n.

$$f_x(0,0) = 2x - 2y + 3x^2 + 5x^4\Big]_{(0,0)} = 0.$$

$$f_y(0,0) = -2x + 2y - 3y^2\Big]_{(0,0)} = 0.$$

$\Rightarrow (0,0)$ is a stationary pt.

$$f_{xx} = 2 + 6x\Big]_{(0,0)} = 2$$

$$f_{yy} = 2 - 6y\Big]_{(0,0)} = 2$$

$$f_{xy} = -2$$

$$H = f_{xx}(0,0)\, f_{yy}(0,0) - \left\{ f_{xy}(0,0) \right\}^2 = 4 - 4 = 0.$$

$$f(x,y) = (x-y)^2 + (x-y)\left\{ \cancel{x^3 - x^2y + xy^2 - y^3}\; x^2 + xy + y^2 \right\} + x^5$$

$= x^5$ along the line $x = y$.

$\Rightarrow \exists$ nbd. of $(0,0)$ containing pts. where $f(x,y) > 0$ & also pts. where

$f(x,y) < 0.$

$\Rightarrow (0,0)$ is a saddle pt.

Example: Show that the f^n.

$$f(x,y) = y^2 + x^2 y + x^4$$

has a minimum at $(0,0)$.

Sol^n. $f_x(0,0) = 0$

$f_y(0,0) = 0$.

~~$f(x,y)$~~ H $\left.\right] = 0$. undecided case at $(0,0)$.

Consider

$$f(x,y) - f(0,0) = y^2 + x^2 y + x^4 - 0$$

$$= \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4.$$

$$> 0 \quad \forall (x,y) \text{ near } (0,0).$$

$\Rightarrow f$ has min^m. at $(0,0)$.

Example: Prove that

$$f(x,y,z) = (x+y+z)^3 - 3(x+y+z) - 24xyz - a^3$$

has max. at $(-1,-1,-1)$ and min. at $(1,1,1)$.

Sol^n.

$$\left.\begin{matrix} f_x(1,1,1) \\ f_y(1,1,1) \\ f_z(1,1,1) \end{matrix}\right\} \quad \left.\begin{matrix} f_x(-1,-1,-1) \\ f_y(-1,-1,-1) \\ f_z(-1,-1,-1) \end{matrix}\right\} \text{ all } 0?$$

(i) $(1,1,1)$

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{pmatrix}$$

principal minors

$$18, \quad \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288, \quad |H| = 16 \times 216.$$

all +ve

So min^{m}. at $(1,1,1)$

(ii) $(-1,-1,-1)$

$$H = \begin{pmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{pmatrix}$$

principal minors

$$-18, \quad \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288, \quad |H| = -16 \times 216.$$

alternatively -ve & +ve.

So max^{m}. at $(-1,-1,-1)$.