Differential Calculus

25.7.16

Rolle's Th.

- $f(x) \rightarrow$ defined $[a, b]$.

i) f continuous in $[a, b]$

ii) f differentiable in (a, b).

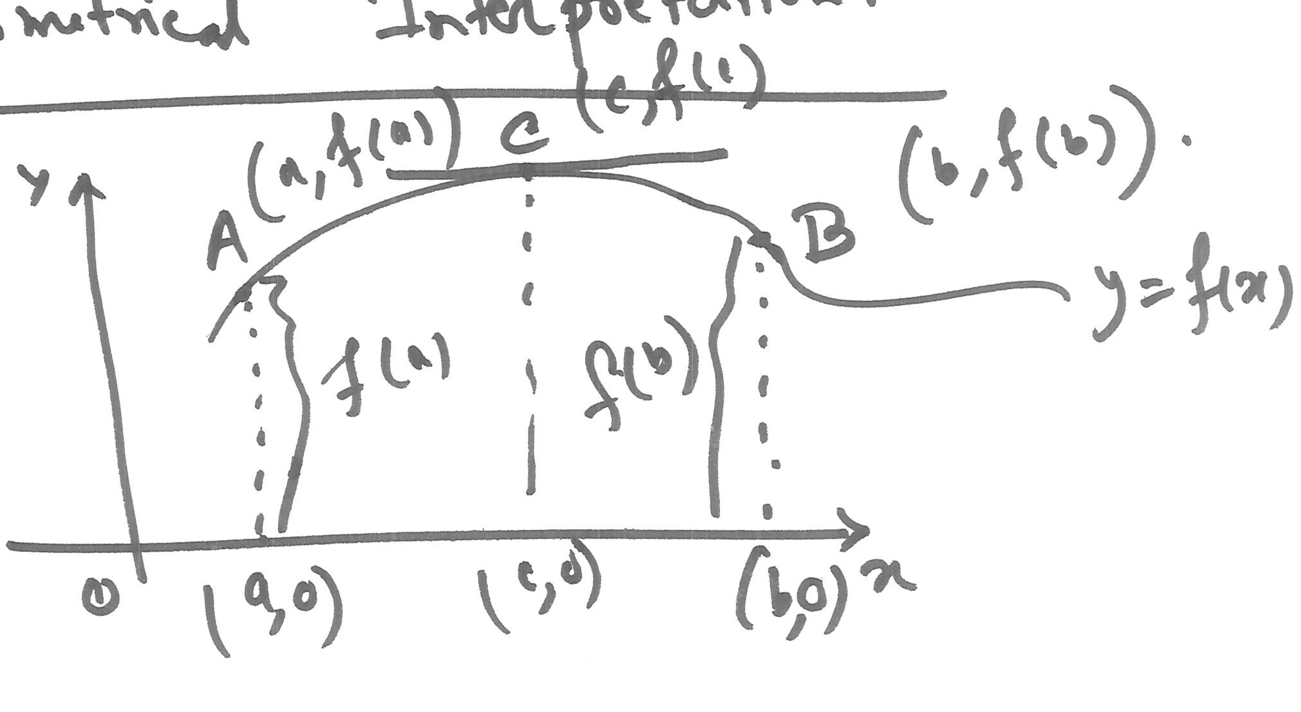
iii) $f(a) = f(b)$.

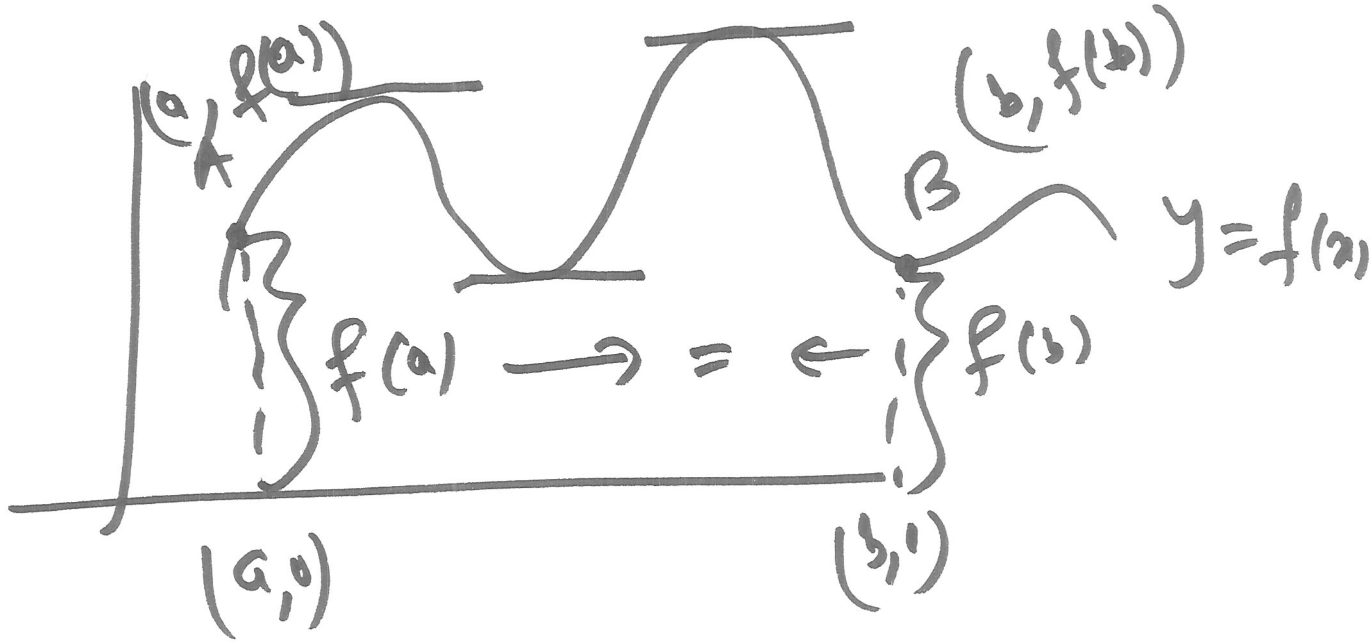
Then Rolle's Th. states that

$\exists$ $\boxed{\text{at least}}$ one value c,

$a < c < b$, s.t. $\boxed{f'(c) = 0}$.

Geometrical Interpretation.





Example:

$f(x) = x\sqrt{a^2 - x^2}$, $[0, a]$.

→ Continuum in $[0, a]$

→ differentiable in $(0, a)$

$$f'(x) = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

→ $f(0) = 0 = f(a)$

By Rolle's Th.

$\Rightarrow$ $\exists$ at least one c,

$0 < c < a$ s.t. $f'(c) = 0$

$\Rightarrow c = a/\sqrt{2}$

Example: $f(x) = \sin x$, $\left[\frac{\pi}{2}, 101\frac{\pi}{2}\right]$.

→ cont. in $\left[\frac{\pi}{2}, 101\frac{\pi}{2}\right]$

→ diff. in $\left(\frac{\pi}{2}, 101\frac{\pi}{2}\right)$.

→ $f\left(\frac{\pi}{2}\right) = 1 = f\left(101\frac{\pi}{2}\right)$

⇒ By Rolle's Th., $\exists$ at least one c, $\frac{\pi}{2} < c < 101\frac{\pi}{2}$ s.t.

$f'(c) = 0$.

↳ many such c exists in $\left(\frac{\pi}{2}, 101\frac{\pi}{2}\right)$

Corollary: a, b → two roots of equ^n $f(x) = 0$,

then $f'(x) = 0$ will have at least one root in (a, b);

provided

i) $f(x)$ is continuous in $[a,b]$

ii) $f'(x)$ exists in (a,b).

<u>Example:</u> $f(x) \rightarrow$ poly.

<u>Example:</u> Show that the eqn.

$3^x + 4^x = 5^x$ has <u>exactly</u> one real root using Rolle's Th.

<u>Soln.</u>

$$f(x) = \left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x - 1$$

$$\Rightarrow \boxed{f(2) = 0}$$

$$f'(x) = \left(\frac{3}{5}\right)^x \log\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)^x \log\left(\frac{4}{5}\right)$$

< 0 for all real x.

• Conditions of the Rolle's Th. are only sufficient, by no means necessary.

Example:

$$f(x) = |x| \text{ in } [-1, 1].$$

$$f(x) = \begin{cases} -x, & -1 \le x < 0 \\ 0, & x = 0 \\ x, & 0 < x \le 1. \end{cases}$$

$f(x) \to$ continuous in $[-1, 1]$.

$\to$ not differentiable at $x = 0$

$\to f(-1) = f(1)$

$\Rightarrow$ Rolle's Th. condition violated.

Conclusion of the Rolle's Th. may or may not hold.

Example: $f(x) = \frac{1}{x} + \frac{1}{1-x}$ in $[0,1]$.

$\mapsto$ not continuous in $[0,1]$.

at $x=0$, $x=1$ discontinuous.

Conclusion of the <u>Rolle's Th.</u> may or may not be true

$$f'(x) = -\frac{1}{x^2} + \frac{1}{(1-x)^2}.$$

$$0 < c < 1, \quad f'(c) = 0$$

$$\Downarrow$$

$$c = 1/2$$

Exercise Discuss the applicability of Rolle's Th. to the fun.

$$f(x) = \begin{cases} x^2+1 & \text{when } 0 \le x \le 1 \\ 3-x & \text{when } 1 < x \le 2 \end{cases}$$

Example: Prove that if $a_0, a_1, \ldots, a_n$ are real numbers s.t.

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n = 0,$$

then there exists at least one no. x, $0 < x < 1$ s.t.

$$\boxed{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0.}$$

Sol^n.

$$f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \cdots + \frac{a_{n-1}}{2} x^2 + a_n x,$$

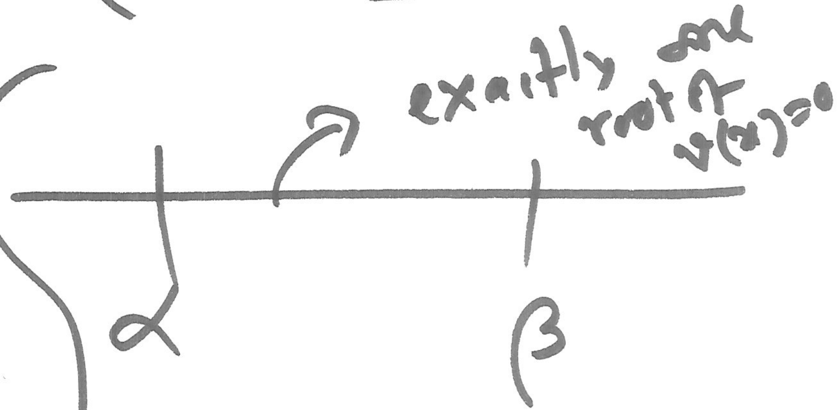
$[0, 1]$

$f(0) = 0 = f(1)$

$\Rightarrow$ satisfies all the three conditions of Rolle's Th.

$\Rightarrow \exists$ at least one pt. c, $0 < c < 1$, s.t. $f'(c) = 0 \Rightarrow$ gives the reqd. result

Example: In any interval in which the fn.s $u(x)$, $v(x)$, $u'(x)$, $v'(x)$ are continuous & $\boxed{uv' - u'v \neq 0}$, the roots of $u(x)$ & $v(x)$ separates each other.



Soln.

$\alpha, \beta \to$ two consecutive roots of $u(x) = 0$

$u(\alpha) = 0 = u(\beta)$.

let $\boxed{f(x) = \dfrac{u(x)}{v(x)}}$, $[\alpha, \beta]$.

$f(\alpha) = 0 = f(\beta)$

Claim $v(x) = 0$ has exactly one root in between $[\alpha, \beta]$.

if not, then $v(x) \neq 0$ in $[\alpha, \beta]$.

$f(x) = \frac{u(x)}{v(x)}$

→ Continuous $[\alpha, \beta]$

→ diff. (α, β)

$f'(x) = \frac{u'v - uv'}{v^2}$, in (α, β).

→ $f(\alpha) = f(\beta)$

⇒ all the conditions of Rolle's Th. are satisfied.

$\therefore \exists \ c \in (\alpha, \beta)$ s.t.

$f'(c) = 0$

$\Rightarrow \left. u'v - uv' \right]_{x=c} = 0$ $(\rightarrow\leftarrow)$

$\therefore \ v(x) = 0$ in $[\alpha, \beta]$.

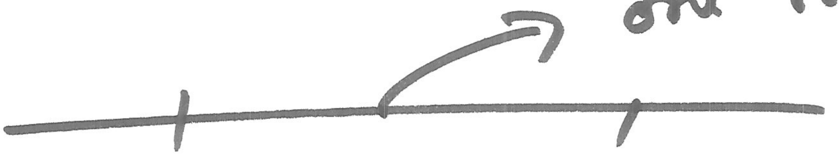
<u>To prove</u> $v(x)=0$ has exactly one root in between $[\alpha, \beta]$.

one root of $v(x)=0$

α $\qquad$ β.

$u(\alpha)=0$ $\qquad$ $u(\beta)=0$, α, β consecutive.

one root of $u(x)=0$.



α' $\qquad$ β'

$v(\alpha')=0$ $\qquad$ $v(\beta')=0$, α', β' consecutive.

Mean Value Theorem.

(Lagrange's form).

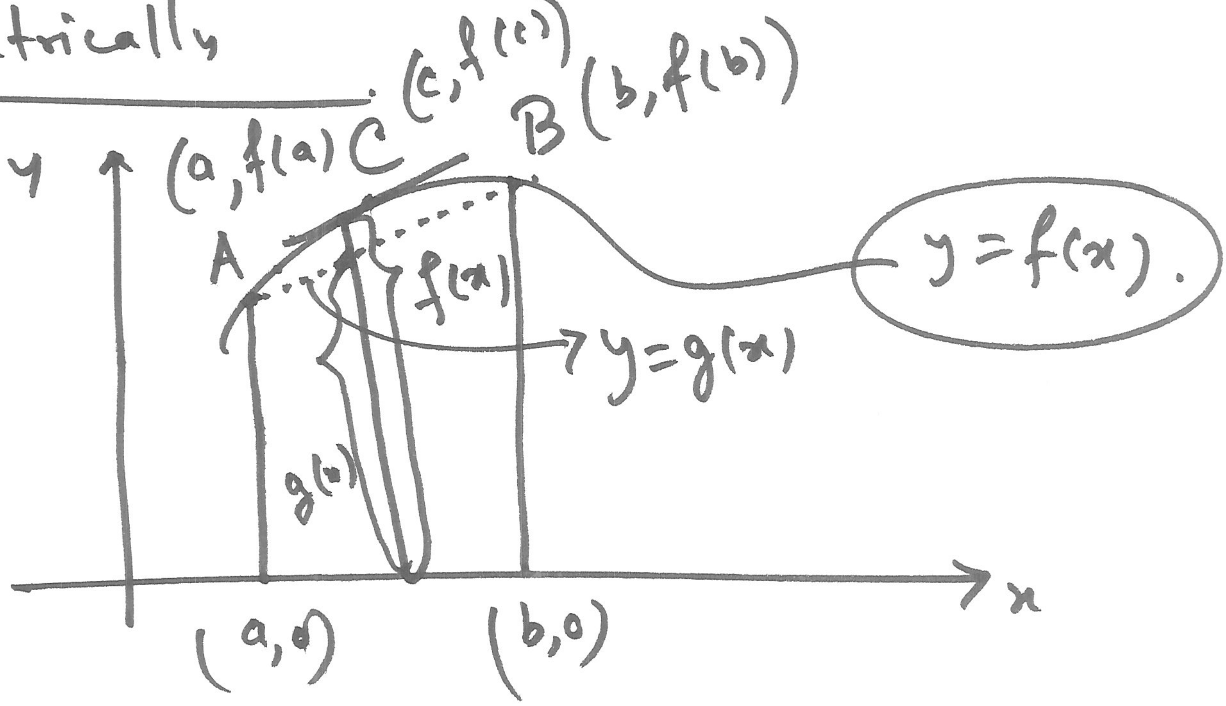
- If a fn. f is
  - i) Continuous in $[a, b]$
  - ii) derivable in (a, b).

then $\exists$ <u>at least</u> one value c, $a<c<b$,

s.t.

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c)}$$

Geometrically



~~eqn.~~

proof: eqn. of the chord AB

$$y = g(x)$$

$$\frac{g(x) - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}.$$

$$\Rightarrow g(x) = f(a) + \frac{(x-a)}{(b-a)}\left(f(b) - f(a)\right)$$

$$h(x) = f(x) - g(x).$$

$$= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x-a)$$

- → continuous in $[a,b]$
- → diff. in (a,b)
- → $h(a) = 0$, $h(b) = 0$

$\therefore$ By Rolle's Th., $\exists$ $\boxed{\text{at least}}$ one $c \in (a,b)$ s.t. $\cancel{f'(c)} \cancel{=} 0.$

$$\boxed{h'(c) = 0.}$$

$$f'(c) - \frac{f(b) - f(a)}{b-a} = 0.$$

$$\Rightarrow \boxed{\frac{f(b) - f(a)}{b-a} = f'(c), \quad a < c < b}$$

$\to$ MVTh.

- Conditions of the MVTh. are <u>only sufficient</u>, by <u>no means</u> <u>Necessary</u>

<u>Example:</u> $f \to$ real valued fn^n. defined over $[-1, 1]$

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

Does MVTh. hold for f in $[-1,1]$?

Sol^n. Check continuity

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = 0. \\ = f(0)$$

$\Rightarrow$ Continuous in $[-1,1]$ ✓

Differentiability check. at $x=0$.

$$f'(0) = \lim_{x\to 0} \frac{f(0+x)-f(0)}{x}$$

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0+} \frac{x\sin\frac{1}{x}-0}{x}$$

$$= \lim_{x\to 0+} \sin\frac{1}{x}$$

does not exist.

$\Rightarrow f$ is not derivable at $x=0$.

$\therefore$ Conclusion of MVTh. may or may not hold.

h-θ form of MVTh.

- $$\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}, \quad a<c<b. \quad [a,b]$$

b = a+h $\quad [a, a+h]$.

$c = a+\theta h, \quad 0<\theta<1.$

The theorem takes the following form:

- f continuous in $[a, a+h]$
- f diff. in $(a, a+h)$.

$\exists$ a no. θ, $\underline{0<\theta<1}$ for which

$$\underline{f(a+h) = f(a) + h f'(a+\theta h)}.$$

(Lagrange's MVTh.).

Example: $f(x) = px^2 + qx + r$, $[a, a+h]$

Find θ of Lagrange's MVT

Ans. $\theta = \frac{1}{2}$, whatever p, q, r, a, h may be.

· Maclaurin's formula.

$[a, a+h]$
$\downarrow$
$[0, x]$

$h = x$, $a = 0$.

$$\boxed{f(x) = f(0) + x f'(\theta x), \quad 0 < \theta < 1}$$

Example: $f(x) = \frac{1}{|x|}$, $[a, b]$

$a = -1$, $b \geq 1$.

$f(x)$ is not defined at $x = 0$ when $0 \in [a, b]$.

$\Rightarrow$ MVTh. conclusion may or may not be true.

$$\boxed{f(x) = \begin{cases} \dfrac{1}{|x|} & , x \neq 0 \\ 3 & , x = 0. \end{cases}}$$

$f(x) = \dfrac{1}{|x|}$, $[-1, b]$, $b \geq 1$

$f(-1) = 1$, $f(b) = \dfrac{1}{b}$

$$\frac{f(b) - f(-1)}{b - (-1)} = \frac{\frac{1}{b} - 1}{b+1}.$$

The ~~cont~~ conclusion of MVTh is true iff $b > 1 + \sqrt{2}$

<u>At $x = 0$</u>

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist.

$$f'(x) = \begin{cases} -\dfrac{1}{x^2}, & x > 0 \\[2ex] \dfrac{1}{x^2}, & -1 < x < 0. \end{cases}$$

$$\boxed{\frac{f(b)-f(-1)}{b-(-1)} = f'(c)}$$

$$\Downarrow$$

$$\frac{\frac{1}{b}-1}{b+1} = \frac{1-b}{b(b+1)} = \begin{cases} -\dfrac{1}{c^2}, & c > 0. \\[2ex] \dfrac{1}{c^2}, & -1 < c < 0. \\ \times & \end{cases}$$

$$b \geq 1.$$

$$\frac{1-b}{b(b+1)} = -\frac{1}{c^2}, \quad \underline{-1 < c < b.}$$

$$\frac{b(b+1)}{-1+b} = c^2 < b^2$$

$$\Rightarrow \quad b+1 < b(-1+b) = b^2 - b.$$

$$b^2 - 2b - 1 > 0.$$

$$(b-1)^2 > 2.$$

$$\Rightarrow \quad b > 1 \pm \sqrt{2}.$$

as $b > 1$

$$b > 1 + \sqrt{2}$$