WEEK 11 : Lecture Notes

Deterministic PDA:

A PDA $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ is deterministic if

i) $\delta(q, a, z)$ is either empty or a singleton set for each $q \in Q$, $a \in \Sigma \cup \{\varepsilon\}$ and $z \in \Gamma$

ii) $\delta(q, \varepsilon, z) \neq 0$ implies $\delta(q, a, z) = \phi \ \forall \ a \in \Sigma$

- DFA, NFA are equivalent with respect to the languages accepted
- Same is not true for PDA
- ww^R is accepted by non-deterministic PDA, but is not accepted by any deterministic PDA

Example

Consider the PDA

$M = (\{q_0, q_1, q_f\}, \{a, b, z_0\}, \delta, q_0, z_0, \{q_f\})$

where δ is defined by:

$\delta(q_0, a, z_0) = \{(q_0, az_0)\}$, $\delta(q_0, b, z_0) = \{(q_0, bz_0)\}$

$\delta(q_0, a, a) = \{q_0, aa\}$, $\delta(q_0, b, a) = \{(q_0, ba)\}$

$\delta(q_0, a, b) = \{q_0, ab\}$, $\delta(q_0, b, b) = \{(q_0, bb)\}$

$\delta(q_0, c, a) = \{(q_1, a)\}$, $\delta(q_0, c, b) = \{(q_1, b)\}$

$\delta(q_0, c, z_0) = \{(q_1, z_0)\}$, $\delta(q_1, a, a) = \delta(q_1, b, b) = \{(q_1, \varepsilon)\}$

$\delta(q_1, \varepsilon, z_0) = \{(q_f, z_0)\}$

$\delta(q_f, \varepsilon, z_0) = \{(q_f, \varepsilon)\}$

Then,

$$N(M) = \{wcw^R \mid w \in \{a, b\}^*\}$$

Pushdown Automata and Context-free languages

- Class of languages accepted by PDA's are precisely the class of CFLs.

    - language accepted by PDAs final state are exactly the languages accepted by PDA's by empty stack, i.e. $N(M_1) = L$ iff $L(M_2) = L$, where M_1, M_2 are accepted

    - languages accepted by empty state are exactly the CFLs i.e. $N(M) = L$ iff L is a CFL

Equivalence of acceptance by final state and empty state

Theorem:

If L is $L(M_2)$ for some PDA M_2 then L is $N(M_1)$ for some PDA M_1

Proof:

Let $M_2 = (Q, \Sigma, \Gamma, \delta, q_0, z_0, F)$ be a PDA

such that $L = L(M_2)$

Let $M_1 = (Q \cup \{q_e, q_0'\}, \Sigma, \Gamma \cup \{X_0\}, \delta', q_0,$

where δ' is defined as follows:

R_1: $\delta'(q_0', \varepsilon, X_0) = \{(q_0, z_0 X_0)\}$

R_2: $\delta'(q, a, z) = \delta(q, a, z) \ \forall (q, a, z) \in Q \times \{\Sigma \cup \{\varepsilon\}\} \times \Gamma$

R_3: $\delta'(q, \varepsilon, z)$ contains $(q_e, \varepsilon) \ \forall q \in F$ and $z \in \Gamma \cup \{X_0\}$

R_4: $\delta'(q_e, \varepsilon, z)$ contains $(q_e, \varepsilon) \ \forall z \in \Gamma \cup \{X_0\}$

- R_1 causes M_1 to enter the initial ID of M_2, except that M_1 will have its own bottom of the stack marker which is below the symbol of M_2's stack.
- R_2 allow M_1 to simulate M_2
- Should M_2 ever enter a final state R_3 and R_4 allow M_1 the choice of entering state q_e and erasing its stack

Note: M_2 may possibly erase its entire stack for some input x not in $L(M_2)$ (i.e. without entering to a final state of F)

This is why M_1 has its own special bottom-of-stack marker X_0

Otherwise, M_1 in simulating M_2 would also erase its entire stack, thereby accepting x when it should not.

To prove $L(M_2) \subseteq N(M_1)$

Let $x \in L(M_2)$

Then $(q_0, x, z_0) \overset{*}{\underset{M_2}{\vdash}} (q, \varepsilon, \gamma)$ for some $q \in F$

Now consider M_1 with input x

By R_1

$$(q_0', x, X_0) \underset{M_1}{\vdash} (q_0, x, z_0 X_0)$$

By R_2, every move of M_2 is a legal move of M_1, thus

$$(q_0, x, z_0) \overset{*}{\underset{M_1}{\vdash}} (q, \varepsilon, \gamma)$$

and so

$$(q_0', x, X_0) \vdash_{M_1} (q_0, x, Z_0 X_0)$$

$$\vdash^{*}_{M_1} (q, \varepsilon, \gamma X_0)$$

$$\vdash^{*}_{M_1} (q_e, \varepsilon, \varepsilon)$$

By R_3 and R_4

and M_1 accepts x by empty stack.

Thus $x \in N(M_1)$

To prove $N(M_1) \subseteq L(M_2)$

If M_1 accepts x by empty stack, it is easy to show that the sequence of moves must be

i. one move by R_1

ii. then a sequence of moves by R_2 in which M_1 simulates acceptance of x by M_2
followed by the erasing of M_1's stack using R_3 and R_4

Thus $x \in L(M_2)$

Theorem

If L is $N(M_1)$ for some PDA M_1 then L is $L(M_2)$ for some PDA M_2

Proof:

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi)$ be a PDA such that

$$L = N(M_1)$$

Let $M_2 = (Q \cup \{q_0', q_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta', q_0', X_0, \{q_f\})$

where δ' is defined as:

R_1: $\delta'(q_0', \varepsilon, X_0) = \{(q_0, z_0 X_0)\}$

R_2: $\delta'(q, a, z) = \delta(q, a, z) \ \forall\ q \in Q, a \in \Sigma \cup \{\varepsilon\}, z \in \Gamma$

R_3: $\delta'(q, \varepsilon, X_0)$ contains (q_f, ε) $\forall\ q \in Q$

- R_1 causes M_2 to enter the initial ID of M_1, except that M_2 will have its own bottom-of-stack marker X_0
- R_2 allows M_2 to simulate M_1

  Should M_1 ever erase its entire stack, then M_2, when simulating M_1 will erase it's entire stack except the symbol X_0 at the bottom.

- R_3 causes M_2, when X_0 appears, to enter a final state thereby accepting the input x

The proof that $L(M_2) = N(M_1)$ is similar to the previous proof.

Example: Consider the PDA M_1 given by

$$M_1 = (\{q_0, q_1\}, \{a, b\}, \{a, z_0\}, \delta, q_0, z_0, \phi)$$

where δ is given by

R_1: $\delta(q_0, a, z_0) = \{(q_0, az_0)\}$

R_2: $\delta(q_0, a, a) = \{(q_0, aa)\}$

R_3: $\delta(q_0, b, a) = \{(q_1, \varepsilon)\}$

R_4: $\delta(q_1, b, a) = \{(q_1, \varepsilon)\}$

R_5: $\delta(q_1, \varepsilon, z_0) = \{(q_1, \varepsilon)\}$

Determine $N(M_1)$. Also construct a PDA M_2 such that $L(M_2) = N(M_1)$

Soln:

- R_1 stores a in PDS
  R_2 repeatedly stores a in PDS (thus string a^n in PDS)
- R_3 is used to erase a from PDS when b is encountered for the first time and PDA transits to state q
- R_4 repeatedly erases a from PDS when b is encountered and PDA remain in the same state q
- R_5 empties PDS if z_0 remains in stack after processing the entire input.

Thus, $(q_0, a^n b^n, z_0) \vdash^* (q_0, b^n, a^n z_0)$

$\vdash (q_1, b^{n-1}, a^{n-1} z_0)$ by R_1 and R_2

$\vdash^* (q_1, \varepsilon, z_0)$ by R_3 and R_4

$\vdash (q_1, \varepsilon\, \varepsilon)$ by R_5

Thus $a^n b^n \in N(M)$

i.e. $\{a^n b^n \mid n \geq 1\} \subseteq N(M)$

To prove $N(M) \subseteq \{a^n b^n \mid n \geq 1\}$

- Let $w \in N(M)$

$\Rightarrow$ Then $(q_0, w, z_0) \vdash^*_M (q_1, \varepsilon, \varepsilon)$

(Note: PDS can be emptied only when M is in state q_1)

- Also w must start with a, otherwise we cannot make any move.
- We store a in PDS if current input symbol is a and the topmost symbol of PDS is either a or z_0
- PDA erases a in PDS if current input symbol is b
- PDA enters ID $(q_1, \varepsilon, \varepsilon)$ only by R_5
- PDA can reach ID (q_1, ε, z_0) only by erasing a's in PDS, which is possible.
- When # of b's = # of a's and so $w = a^n b^n$

Thus $N(M) = \{a^n b^n \mid n \geq 1\}$

Construction of M_2 such that $L(M_2) = N(M_1)$

$$M_1 = (\{q_0, q_1\}, \{a, b\}, \{a, z_0\}, \delta, q_0, z_0, \phi)$$

$\big\downarrow$ construct

$$M_2 = (\{q_0, q_1, q_0', q_f\}, \{ab\}, \{a, z_0, X_0\}, \delta', q_0', X_0, \{q_f\})$$

where δ' is defined by

added rule → $\delta'(q_0', \varepsilon, X_0) = \{(q_0, z_0X_0)\}$

follow given rules

$$\begin{cases} \delta'(q_0, a, z_0) = \delta(q_0, a, z_0) = \{(q_0, az_0)\} \\ \delta'(q_0, a, a) = \{(q_0, aa)\} \\ \delta'(q_0, b, a) = \{(q_1, \varepsilon)\} \\ \delta'(q_1, b, a) = \{(q_1, \varepsilon)\} \\ \delta'(q_1, \varepsilon, z_0) = \{(q_1, \varepsilon)\} \end{cases}$$

added rules

$$\begin{cases} \delta'(q_0, \varepsilon, X_0) = \{(q_f, \varepsilon)\} \\ \delta'(q_1, \varepsilon, X_0) = \{(q_f, \varepsilon)\} \end{cases}$$

Thus

$$L(M_2) = N(M_1)$$

Equivalence of PDA's and CFL's

Theorem:

If L is a CFL, then there exists a PDA M such that $L = N(M)$

Proof:

Assume that $\varepsilon \notin L(G)$

(The construction can be modified if $\varepsilon \in L(G)$)

Let $G = (V, T, P, S)$ be a CFG in GNF generation L

Let $M = (\{q\}, T, V, \delta, q, S, \phi)$

where $\delta(q, a, A)$ contains (q, γ) whenever $A \to a\gamma$ is in P.

Claim:

$S \overset{*}{\Rightarrow} x\alpha$ by a leftmost derivation iff

$$(q, x, S) \overset{*}{\underset{M}{\vdash}} (q, \varepsilon, \alpha)$$

where $x \in T^*$, $\alpha \in V^*$

Example:

Construct a PDA M equivalent to the following CFG: $S \to 0BB$, $B \to 0S \mid 1S \mid 0$

Test whether 010^4 is in $N(M)$.

Soln:

$M = (\{q\}, \{0,1\}, \{S, B\}, \delta, q, S, \phi)$ where δ is defined by

$\delta(q, 0, S) = \{(q, BB)\}$, $\delta(q, 1, B) = \{(q, S)\}$

$\delta(q, 0, B) = \{(q, S), (q, \varepsilon)\}$

So, $(q, 010^4, S) \vdash (q, 10^4, BB)$

$\vdash (q, 0^4, SB)$

$\vdash (q, 0^3, BBB)$

$\vdash (q, 0^2, BB)$

$\vdash (q, 0, B)$

$\vdash (q, \varepsilon, \varepsilon)$

accept

Thus, $010^4 \in N(M)$

Claim:

$$S \overset{*}{\Rightarrow} x\alpha, \quad x \in T^*, \alpha \in V^* \qquad \text{iff} \qquad (q, x, S) \underset{M}{\overset{*}{\vdash}} (q, \varepsilon, \alpha) \qquad \text{— (1)}$$

Proof:

<u>if part</u>

Let $(q, x, S) \overset{i}{\vdash} (q, \varepsilon, \alpha)$ and show by induction on i that $S \overset{*}{\Rightarrow} x\alpha$

<u>Base: $i = 0$:</u> $(q, x, S) \overset{0}{\vdash} (q, \varepsilon, \alpha)$

i.e. $x = \varepsilon$, $\alpha = S$ and

$S \overset{*}{\Rightarrow} S \; (= x\alpha)$ holds

<u>Induction step</u>

Let $i \geq 0$ and $x = ya$, $y \in T^*$, $a \in T$

Consider next to last step:

$$\underbrace{(q, ya, S) \overset{i-1}{\vdash} (q, a, \beta)}_{(q, y, S) \overset{i-1}{\vdash} (q, \varepsilon, \beta)} \vdash (q, \varepsilon, \alpha), \quad \beta \in V^*$$

By induction hypothesis

$$S \overset{*}{\Rightarrow} y\beta, \quad y \in T^*, \ \beta \in V^*$$

Also, the last move

$$(q, a, \beta) \vdash (q, \varepsilon, \alpha)$$

implies

$$\beta = A\gamma \text{ for some } A \in V$$

$$\left(\begin{array}{l} (q, a, A\gamma) \vdash (q, \varepsilon, \eta\gamma) \text{ if } \delta(q, a, A) \text{ contains } (q, \eta) \\ \qquad = (q, \varepsilon, \alpha) \text{ i.e. if } A \to a\eta \text{ is in } P \\ \quad \text{implies } \alpha = \eta\gamma \end{array} \right)$$

Thus, $S \overset{*}{\Rightarrow} y\beta \Rightarrow ya\eta\gamma = ya\alpha = x\alpha$

and we conclude the 'if' part of (1)

only if part:

Let $S \overset{i}{\Rightarrow} x\alpha$, $x \in T^*$, $\alpha \in V^*$ by a leftmost derivation.

We show by induction on i that

$$(q, x, S) \overset{*}{\vdash} (q, \varepsilon, \alpha)$$

Base: $i = 0$

$x = \varepsilon$, $\alpha = S$ as $S \overset{0}{\Rightarrow} x\alpha$

$$\therefore (q, x, S) = (q, \varepsilon, S) \overset{0}{\vdash} (q, \varepsilon, \alpha)$$

Inductive step:

Let $i > 0$ and suppose $S \overset{i-1}{\Rightarrow} yA\gamma \Rightarrow ya\eta\gamma$

where $x = ya$, $\alpha = \eta\gamma$ and $A \to a\eta$ is in P as G is in GNF

By induction hypothesis

$$(q, y, S) \vdash_M^* (q, \varepsilon, A\gamma)$$

and thus $(q, ya, S) \vdash_M^* (q, a, A\gamma)$

Since $A \to a\eta$ is in P, it follows that $\delta(q, a, A)$ contains (q, η).

Thus $(q, x, S) \vdash_M^* (q, a, A\gamma)$

$$\vdash_M (q, \varepsilon, \eta\gamma) = (q, \varepsilon, \alpha)$$

and the 'only if' part of 1 follows.

Alternative construction

If G is not in GNF, set PDA M to be

$$M = (\{q\}, T, V \cup T, \delta, q, S, \phi)$$

where δ is defined as follows:

1. $\delta(q, \varepsilon, A) = \{(q, \alpha) \mid A \to \alpha \text{ is in } P\}$
2. $\delta(q, a, a) = \{(q, \varepsilon)\}$ for every $a \in T$

Example: Convert the CFG G given below to a PDA.

$$E \to I \mid E+E \mid E*E \mid (E)$$

$$I \to a \mid b \mid Ia \mid Ib \mid I0 \mid I1$$

Soln:

$$M = (\{q\}, \underbrace{\{a, b, 0, 1, +, *, (,)\}}_{= T}, \{I, E\} \cup T, \delta, q, S, \phi\}$$

where δ is defined as

$$\delta(q, \varepsilon, E) = \{(q, I), (q, E+E), (q, E*E), (q, (E))\}$$

$$\delta(q, \varepsilon, I) = \{(q, a), (q, b), (q, Ia), (q, Ib), (q, I0), (q, I1)\}$$

$$\delta(q, c, c) = \{(q, \varepsilon)\} \quad \forall\, c \in T$$

Example:

G is a CFG with productions

$S \to 0BB, \quad B \to 0S \mid 1S \mid 0$

Convert it to PDA M such that

$N(M) = L(G)$

Soln:

$M = (\{q\}, \underbrace{\{0,1\}}_{T}, \underbrace{\{S, B, 0, 1\}}_{V \cup T}, \delta, q, S, \phi)$

where δ is defined as

$$\delta(q, \varepsilon, S) = \{(q, 0BB)\}$$
$$\delta(q, \varepsilon, B) = \{(q, 0S), (q, 1S), (q, 0)\}$$

$$\delta(q, 0, 0) = \{(q, \varepsilon)\}$$
$$\delta(q, 1, 1) = \{(q, \varepsilon)\}$$

Check $010^4 \in N(M)$ or not.

$$(q, 010^4, S) \vdash (q, 010^4, 0BB)$$
$$\vdash (q, 10^4, BB)$$
$$\vdash (q, 10^4, 1SB)$$
$$\vdash (q, 0^4, SB)$$
$$\vdash (q, 0^4, 0BBB)$$
$$\vdash (q, 0^3, BBB)$$
$$\vdash (q, 0^3, 0BB)$$
$$\vdash (q, 0^2, BB)$$
$$\vdash (q, 0^2, 0B)$$
$$\vdash (q, 0, B)$$
$$\vdash (q, 0, 0)$$
$$\vdash (q, \varepsilon, \varepsilon) \quad \text{accept}$$

Theorem

The following three statements are equivalent

1. L is a CFL
2. $L = N(M_1)$ for some PDA M_1
3. $L = L(M_2)$ for some PDA M_2

(Proofs skipped, only constructions are given)

- (3) $\Rightarrow$ (2)

$$M_2 = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$$

such that $L = L(M_2)$

$\Big\downarrow$ construct M_1 s.t. $L = N(M_1)$

$$M_1 = (Q \cup \{q_e, q_0'\}, \Sigma, \Gamma \cup \{X_0\}, \delta', q_0', X_0, \Phi)$$

where δ' is defined as follows

R_1: $\delta'(q_0', \varepsilon, X_0) = \{(q_0, Z_0X_0)\}$

$\delta'(q, a, z) = \delta(q, a, z)\ \forall\, q \in Q,\ a \in \Sigma \cup \{\varepsilon\},\ z \in \Gamma$

$\delta'(q, \varepsilon, z)$ contains $(q_e, \varepsilon)\ \forall\, q \in F,\ z \in \Gamma \cup \{X_0\}$

$\delta'(q_e, \varepsilon, z)$ contains $(q_e, \varepsilon)\ \forall\, z \in \Gamma \cup \{X_0\}$

• (2) $\Rightarrow$ (3)

$M_1 = (Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi)$ such that $L = N(M_1)$

$\downarrow$ construct M_2 s.t. $L = L(M_2)$

$M_2 = (Q \cup \{q_0', q_f\}, \Sigma, \Gamma \cup \{X_0\}, \delta', q_0', X_0, \{q_f\})$

where δ' is defined as follows:

R_1: $\delta'(q_0', \varepsilon, X_0) = \{(q_0, z_0 X_0)\}$

R_2: $\delta'(q, a, z) = \delta(q, a, z)$ $\forall q \in Q, a \in \Sigma \cup \{\varepsilon\}$ and $z \in \Gamma$

R_3: $\delta'(q, \varepsilon, X_0)$ contains (q_f, ε) $\forall q \in Q$

• (1) $\Rightarrow$ (2)

$G = (V, T, P, S)$ such that $L = L(G)$

$\downarrow$ construct M_1 s.t. $L = N(M_1)$

$M_1 = (\{q\}, T, V, \delta, q, S, \phi)$

where $\delta(q, a, A)$ contains (q, r)

whenever $A \to ar$ is in P.

Alternative construction of M_1 from G

$G = (V, T, P, S)$ not in GNF with $L = L(G)$

$\downarrow$ construct M_1 s.t. $L = N(M_1)$

$M_1 = (\{q\}, T, V \cup T, \delta, q, S, \phi)$

where δ is defined as

(i) $\delta(q, \varepsilon, A) = \{(q, \alpha) \mid A \to \alpha \text{ is in } P\}$

(ii) $\delta(q, a, a) = \{(q, \varepsilon)\}$ for every $a \in T$

• (2) $\Rightarrow$ (1)

$M_1 = (Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi)$ such that $L = N(M_1)$

$\downarrow$ construct $G = (V, T, P, S)$ s.t. $L = L(G)$

$G = (V, T, P, S)$, $T = \Sigma$, $V = S \cup \{[q, z, p] \mid q, p \in Q, z \in \Gamma\}$

and P is set of following productions

i. $S \rightarrow [q_0, z_0, p] \;\forall\; p \in Q$

ii. $(q_1, \varepsilon) \in \delta(q, a, z)$ induces production

$$[q, z, q_1] \rightarrow a$$

iii. $(q_1, z_1 z_2 \ldots z_m) \in \delta(q, a, z)$ yields productions

$$[q, z, q_{m+1}] \rightarrow a\,[q_1, z_1, q_2]\,[q_2, z_2, q_3] \ldots \ldots [q_m, z_m, q_{m+1}]$$

for each $q_1, q_2 \ldots q_{m+1} \in Q$

$z_1, z_2, \ldots z_m \in \Gamma$

and $a \in \Sigma \cup \{\varepsilon\}$

Theorem

If L is $N(M)$ for some PDA M, then L is a CFL.

Proof:

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, z_0, \phi)$

(construction of G)

We define $G = (V, \Sigma, P, S)$ where

$V = \{S\} \cup \{ [q, z, p] \mid q, p \in Q,\ z \in \Gamma \}$

P is the set of productions:

1. $S \to [q_0, z_0, q]$ for each $q \in Q$

2a. $[q, z, q_{m+1}] \to a [q_1, z_1, q_2][q_2, z_2, q_3] \dots [q_m, z_m, q_{m+1}]$

for each $q, q_1, q_2, \dots q_{m+1} \in Q$

each $a \in \Sigma \cup \{\varepsilon\}$ and

$z_1, z_2, \dots z_m \in \Gamma$ such that

$(q_1, z_1 z_2 \dots z_m) \in \delta(q, a, z)$

2b. if $m = 0$, then the production is

$[q, z, q_1] \to a$ induced by $(q_1, \varepsilon) \in \delta(q, a, z)$

corresponding to move erasing a pushdown symbol

$\therefore \quad L(G) = N(M)$

Example: Construct a CFG G which accepts $N(M)$ where

$$M = (\{q_0, q_1\}, \{a, b\}, \{z_0, z\}, \delta, q_0, z_0, \phi)$$

and δ is given by

$$\delta(q_0, b, z_0) = \{(q_0, zz_0)\}$$
$$\delta(q_0, \varepsilon, z_0) = \{(q_0, z)\}$$
$$\delta(q_0, b, z) = \{(q_0, zz)\}$$
$$\delta(q_0, a, z) = \{(q_1, z)\}$$
$$\delta(q_1, b, z) = \{(q_1, \varepsilon)\}$$
$$\delta(q_1, a, z_0) = \{(q_0, z_0)\}$$

Sol^n:

Let

$G = (V, T, P, S)$ generating $N(M)$

$T = \{a, b\}$

$V = S \cup \{[q_0, z_0, q_0], [q_0, z, q_0], [q_0, z_0, q_1],$
$[q_0, z, q_1], [q_1, z_0, q_0], [q_1, z, q_0],$
$[q_1, z_0, q_1], [q_1, z, q_1]\}$

The productions are:

S- productions:

1. $S \to [q_0, z_0, q_0] \mid [q_0, z_0, q_1]$

Other productions:

2. $\delta(q_0, b, z_0) = \{(q_0, zz_0)\}$ yields

$$[q_0, z_0, q_0] \to b[q_0, z, q_0][q_0, z_0, q_0]$$
$$[q_0, z_0, q_0] \to b[q_0, z, q_1][q_1, z_0, q_0]$$
$$[q_0, z_0, q_1] \to b[q_0, z, q_0][q_0, z_0, q_1]$$
$$[q_0, z_0, q_1] \to b[q_0, z, q_1][q_1, z_0, q_1]$$

3. $\delta(q_0, \varepsilon, z_0) = \{(q_0, \varepsilon)\}$ yields

$[q_0, z_0, q_0] \rightarrow \varepsilon$

4. $\delta(q_0, b, z) = \{(q_0, zz)\}$ yields

$[q_0, z, q_0] \rightarrow b[q_0, z, q_0][q_0, z, q_0]$

$[q_0, z, q_0] \rightarrow b[q_0, z, q_1][q_1, z, q_0]$

$[q_0, z, q_1] \rightarrow b[q_0, z, q_0][q_0, z, q_1]$

$[q_0, z, q_1] \rightarrow b[q_0, z, q_1][q_1, z, q_1]$

5. $\delta(q_0, a, z) = \{(q_1, z)\}$ yields.

$[q_0, z, q_0] \rightarrow a[q_1, z, q_0]$

$[q_0, z, q_1] \rightarrow a[q_1, z, q_1]$

6. $\delta(q_1, a, z_0) = \{(q_0, z)\}$ yields

$[q_1, z_0, q_0] \rightarrow a[q_0, z_0, q_0]$

$[q_1, z_0, q_1] \rightarrow a[q_0, z_0, q_1]$

7. $\delta(q_1, b, z) = \{(q_1, \varepsilon)\}$ yields

$[q_1, z, q_1] \rightarrow b$

Example:

Construct a CFG G which accepts $N(M)$ where

$$M = (\{q\}, \{0,1\}, \{Z, A, B\}, \delta, q, Z)$$

where δ is defined by

$\delta(q,0,Z) = \{(q, AZ)\}$, $\delta(q,1,Z) = \{(q, BZ)\}$,

$\delta(q,0,A) = \{(q, AA)\}$, $\delta(q,1,A) = \{(q,\varepsilon)\}$,

$\delta(q,0,B) = \{(q,\varepsilon)\}$, $\delta(q,1,B) = \{(q,BB)\}$

$\delta(q,\varepsilon,Z) = \{(q,\varepsilon)\}$

Sol^{n}:

$G = (V, T, P, S)$, $T = \{0,1\}$

$V = S \cup \{[q,Z,q], [q,A,q], [q,B,q]\}$

Productions:

- S- productions:

  $S \to [q,Z,q]$

- $\delta(q,0,Z) = \{(q,AZ)\}$ yields

  $[q,Z,q] \to 0[q,A,q][q,Z,q]$

- $\delta(q,1,Z) = \{(q,BZ)\}$ yields

  $[q,Z,q] \to 1[q,B,q][q,Z,q]$

- $\delta(q,0,A) = \{(q,AA)\}$ yields

  $[q,A,q] \to 0[q,A,q][q,A,q]$

- $\delta(q,1,A) = \{(q,\varepsilon)\}$ yields

  $[q,A,q] \to 1$

- $\delta(q,0,B) = \{(q,\varepsilon)\}$ yields

  $[q,B,q] \to 0$

- $\delta(q, 1, B) = \{(q, BB)\}$ yields

  $[q, B, q] \to 1\,[q, B, q]\,[q, B, q]$

- $\delta(q, \varepsilon, z) = \{(q, \varepsilon)\}$ yields

  $[q, z, q] \to \varepsilon$

Let

$X = [q, z, q]$, $C = [q, A, q]$, $D = [q, B, q]$

Then the productions are:

$S \to X$

$X \to 0CX$

$X \to 1DX$

$C \to 0CC$

$C \to 1$

$D \to 0$

$D \to 1DD$

$X \to \varepsilon$

Thus,

$G = (\{S, C, D\}, \{0, 1\}, \{S \to 0CS \mid 1DS \mid \varepsilon,\ C \to 0CC \mid 1$
$D \to 1DD \mid 0 \mid S)$

Example:

Let $M = (\{q_0, q_1\}, \{0,1\}, \{X, z_0\}, \delta, q_0, z_0, \phi)$

where δ is given by

$\delta(q_0, 0, z_0) = \{(q_0, Xz_0)\}$, $\delta(q_1, 1, X) = \{(q_1, \varepsilon)\}$

$\delta(q_0, 0, X) = \{(q_0, XX)\}$, $\delta(q_1, \varepsilon, X) = \{(q_1, \varepsilon)\}$

$\delta(q_0, 1, X) = \{(q_1, \varepsilon)\}$, $\delta(q_1, \varepsilon, z_0) = \{(q_1, \varepsilon)\}$

To construct a CFG $G = (V, T, P, S)$ generating $N(M)$

Sol$^{\underline{n}}$:

$T = \{0,1\}$

$V = S \cup \{[q_0, z_0, q_0], [q_0, z_0, q_1],$
$[q_0, X, q_0], [q_0, X, q_1],$
$[q_1, z_0, q_0], [q_1, z_0, q_1],$
$[q_1, X, q_0], [q_1, X, q_1]\}$

Productions in P

1. $S \rightarrow [q_0, z_0, q_0] \mid [q_0, z_0, q_1]$

2. $\delta(q_0, 0, z_0) = \{(q_0, X, z_0)\}$ induces

$[q_0, z_0, q_0] \rightarrow 0[q_0, X, q_0][q_0, z_0, q_0]$

$[q_0, z_0, q_0] \rightarrow 0[q_0, X, q_1][q_1, z_0, q_0]$

~~$\delta(q_1, 1, X) = \{(q_1, \varepsilon)\}$ yields~~

$[q_0, z_0, q_1] \rightarrow 0[q_0, X, q_0][q_0, z_0, q_1]$

$[q_0, z_0, q_1] \rightarrow 0[q_0, X, q_1][q_1, z_0, q_1]$

3. $\delta(q_1, 1, X) = \{(q_1, \varepsilon)\}$ induces

$[q_1, X, q_1] \to 1$

4. $\delta(q_0, 0, X) = \{(q_0, XX)\}$ induces

$[q_0, X, q_0] \to 0[q_0, X, q_0][q_0, X, q_0]$

$[q_0, X, q_0] \to 0[q_0, X, q_1][q_1, X, q_0]$

$[q_0, X, q_1] \to 0[q_0, X, q_0][q_0, X, q_1]$

$[q_0, X, q_1] \to 0[q_1, X, q_1][q_1, X, q_1]$

5. $\delta(q_1, \varepsilon, X) = \{(q_1, \varepsilon)\}$ yields

$[q_1, X, q_1] \to \varepsilon$

6. $\delta(q_0, 1, X) = \{(q_1, \varepsilon)\}$ yields

$[q_0, X, q_1] \to 1$

7. $\delta(q_1, \varepsilon, Z_0) = \{(q_1, \varepsilon)\}$ yields

$[q_1, Z_0, q_1] \to \varepsilon$