Process Dynamics and Control

Chemical Engineering

Dynamics

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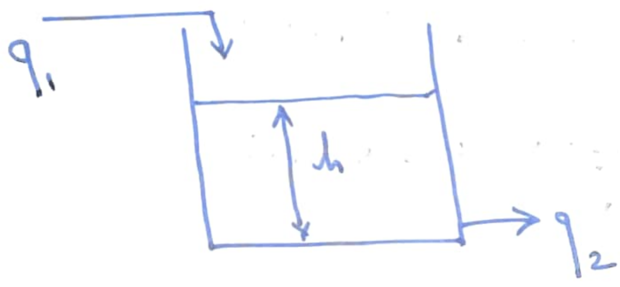
State-space representation of dynamical systems

Following the classical definition of dynamical systems, we will identify such systems as the ones in which the state of the system varies with time. Obviously such systems can be represented by a single or collection of differential equations with time as the independent variable. Some objectives to study such systems include the understanding the temporal behaviour of the systems, their stabilities, and their analysis to potentially develop a control system. Therefore, it is important to understand the nature of solutions of the underlying differential equations. We start with the simplest example of a liquid level tank system shown below.



q_1 and q_2 are volumetric flow rates at the inlet and outlet of the tank, respectively. It can be easily realized that the level of liquid, h, in the tank is the dynamical variable. The conservation equation for the above case can be written as an ordinary differential equation as:

$$A\frac{dh}{dt} = q_1 - q_2 \qquad -(1)$$

where A is the constant area of cross-section of the tank. For equal inlet and outlet flowrates $q_1 = q_2$, Eq^n (1) has a constant solution signifying that the level of the liquid in the tank remains constant. The situation becomes quickly complex as soon as we start analysing q_1 and q_2.

Case 1: constant q_1, constant q_2; $q_1 \neq q_2$

$$\frac{dh}{dt} = \frac{q_1 - q_2}{A} = \text{constant}$$

Hence, h will either increase or decrease depending upon whether $q_1 > q_2$ or $q_2 > q_1$. The variation will be linear in time.

Case 2: constant q_1, $q_2 = q_2(h)$

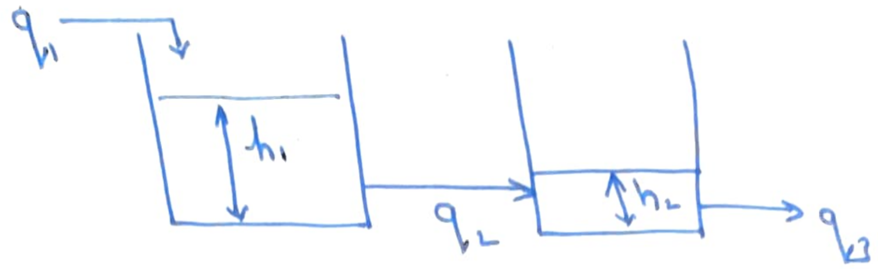
$$\frac{dh}{dt} + \frac{q_2(h)}{A} = \text{constant}$$

Case 3: $q_1 = q_1(t)$, $q_2 = q_2(h)$

$$\frac{dh}{dt} + \frac{q_2(h)}{A} = \frac{q_1(t)}{A}$$

From all of the above situations, we see that the evolution of $h(t)$ depends upon the functions $q_1(t)$ and $q_2(h)$. Hence, it is important understand the behaviour of Eq^n (1) for different q_1 and q_2.

Now let us consider a system having two tanks with the inlet and outlet flowrates and liquid levels as shown below:



The conservation equations can be written as follows:

$$A_1 \frac{dh_1}{dt} = q_1 - q_2 \qquad - (2)$$

$$A_2 \frac{dh_2}{dt} = q_2 - q_3 \qquad - (3)$$

Now in this case, we have a system of two ordinary differential equations which are coupled via the flowrate q_2. Again, the nature of solutions will depend upon q_1, q_2 and q_3. Several possibilities exists as given below.

q_1, q_2, q_3 – constants

q_1 = constant ; $q_1 = q_1(t)$

$q_2 = q_2(h_1)$; $q_2 = q_2(h_1, h_2)$

$q_3 = q_3(h_2)$

Hence, it is essential to know the nature of solutions of Eq^{n}s (2) and (3) to know the temporal behaviour of the system.

The previous case of two-tank problem can be extended to an N-tank problem for which we will get N first order ordinary differential equations. Also, one may have a situation where the governing equation itself may be of a higher order ordinary differential equation. Consider the following case.

$$a_2 \frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y + k = 0 \qquad -(4)$$

Let $$\frac{dy}{dt} = z \qquad -(5)$$

$$\Rightarrow a_2 \frac{dz}{dt} + a_1 z + a_0 y + k = 0 \qquad -(6)$$

It can be seen that a second order ODE, given by Eqn (4), has been converted to a set of two first order ODE's, given by Eqn's (5) & (6). As an extension, we see that an Nth order ODE can be written as a set of N-first order ODE's. Such a representation of a system is called the state-space representation. The state vector in the above case is $[y, z]^T$ and it can be seen that this vector is a dynamical variable whose temporal evolution in time is desirable. Therefore, one needs to model the system and needs to analyze the coupled first order ODE's to know the temporal behaviour of the system. We do this in detail in the following discussion.

We first focus on a system which can be described using N first order linear ordinary differential equations. Let us consider first the homogeneous problem given as follows:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N$$

$$\vdots$$

$$\frac{dx_N}{dt} = a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N$$

The above set of coupled ODE's can be written as a matrix equation as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \quad -(7)$$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}}\,\underline{x} \quad -(8)$$

$\underline{\underline{A}}$ is an $N \times N$ matrix. Therefore, it is possible to determine its eigenvalues and corresponding eigenvectors. Let $\underline{v}$ be any eigenvector of $\underline{\underline{A}}$ with an eigenvalue of λ_v. Consider a vector

$$\underline{x}(t) = e^{\lambda_v t}\,\underline{v}$$

$$\Rightarrow \frac{d}{dt}(\underline{x}(t)) = \frac{d}{dt}(e^{\lambda_v t}\,\underline{v})$$

$$\Rightarrow \frac{d}{dt}(\underline{x}(t)) = e^{\lambda_v t}(\lambda_v \underline{v})$$

Since $\underline{v}$ is an eigenvector with an eigenvalue of λ_v,

$\lambda_u \underline{v} = \underline{\underline{A}}\,\underline{v}$. Using this result, we can write

$$\frac{d}{dt}[X(t)] = e^{\lambda_u t}(\underline{\underline{A}}\,\underline{v})$$

$$\Rightarrow \frac{d}{dt}\left[e^{\lambda_u t}(\underline{v})\right] = \underline{\underline{A}}\left(e^{\lambda_u t}\underline{v}\right)$$

$\Rightarrow$ $e^{\lambda u}\underline{v}$ is a solution to the matrix equation (8). Since $\underline{v}$ is any eigenvector, the above result holds true for any and every eigenvector of $\underline{\underline{A}}$. Hence, following the linearity principle,

$$\underline{X} = \sum_{i=1}^{N} c_i e^{\lambda_i t}\underline{v}_i \qquad -(9)$$

where $\underline{v}_i$'s are the eigenvectors of $\underline{\underline{A}}$ with corresponding eigenvalues λ_i.

Some characteristics of the systems described by Eq^n (8) are worth mentioning here.

- It can be easily seen that Eq^n (8) is a linear equation. This enables us to use linearity principle and write the general solution as Eq^n (9).
- A general matrix equation of the form $\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b}$ can be written where $\underline{b} = \underline{b}(t)$. In this case of $\underline{X}' = \underline{\underline{A}}\,\underline{X}$, $\underline{b}(t) = \underline{0}$ and the problem is referred to as the homogeneous problem.
- It can be seen that the RHS of Eq^n (8) is a function of dependent vector $\underline{x}$ only and is independent of independent variable t. Such an equation is referred to as an autonomous equation.

We start with the simplest case of a first order ODE which is linear and homogeneous and autonomous. The case is given by the following equation.

$$\frac{dx}{dt} = ax \qquad -(10)$$

The above equation arises in case of a single liquid level tank with $q_1 = 0$ and q_2 a linear function of h. 'a' in Eq^n (10) is a parameter. For each value of 'a', we have a different differential equation. It is easy to figure out that the solution of Eq^n (10) is given as

$$x(t) = ke^{at} \qquad -(11)$$

Eq^n (11) represents a collection of solutions and the collection of solutions is referred to as the general solution. The constant 'k' is completely defined if if an initial value of x, $x(0)$, is given. Hence, the problem of Eq^n (11) is an initial value problem. If $x(0) = x_0$ then

$$x(t) = x_0 e^{at}$$

which is a particular solution to ODE with the given initial condition.

When $x_0 = 0$, Eq^n (11) gives $x(t) = 0$. Hence $x = 0$ $\forall\, t$. This means that with time, the value of x will remain constant. A constant solution such as this is called an equilibrium solution. Such solutions are of importance as they are time independent solutions and give important information on the state of the system.

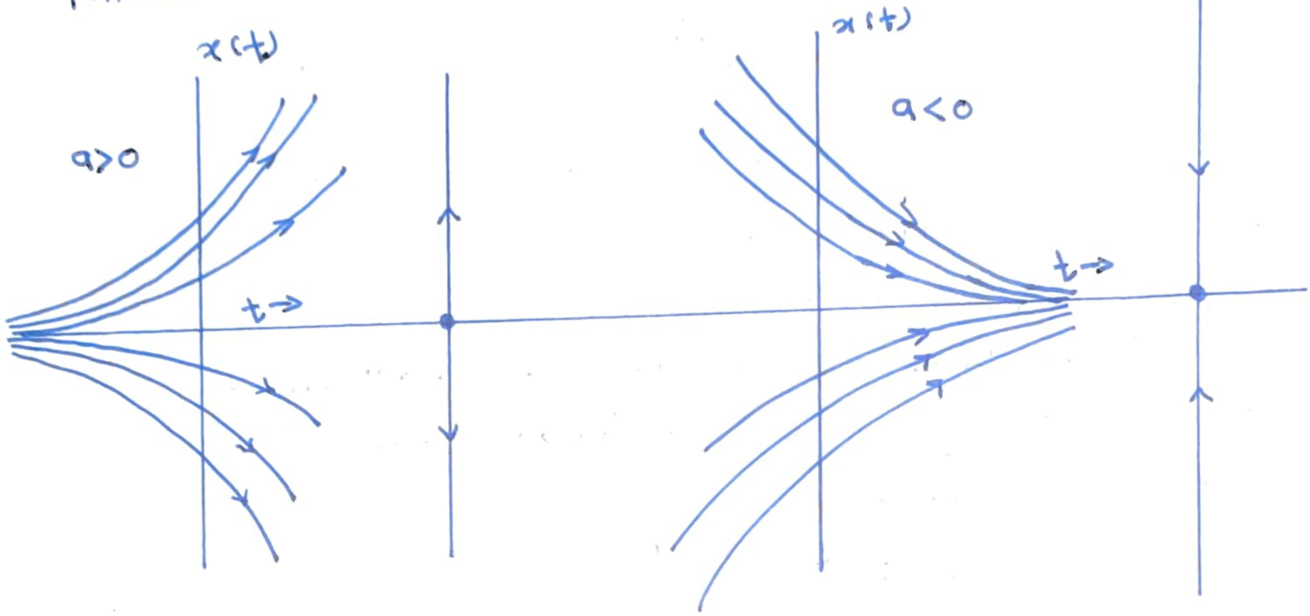
The nature of solution of Eq^n (10) depends upon the value of the parameter 'a'. From Eq^n (11) we can see that

(a) $\lim_{t\to\infty} x_0 e^{at} \to \infty$ for $a > 0$

(b) $\lim_{t\to\infty} x_0 e^{at} = 0$ for $a < 0$

(c) $\lim_{t\to\infty} x_0 e^{at} = x_0 = \text{constant}$ for $a = 0$

The situation can be graphically represented as follows:



The above diagrams are called phase portraits. They represent every possible solution of the equations. Each curve on the portrait is the solution to Eq^n (10) with a different value of 'a'. As seen previously, $x = 0$ is a solution to the equation and this is an equilibrium solution of the system. The non-equilibrium solutions either move towards or move away from the equilibrium solution as $t \to \infty$.

In the current case, when $a > 0$, the solutions tend to move away from the equilibrium solution while for $a < 0$, the solutions tend to move towards the equilibrium solution. The arrows in the phase portrait show the direction of time. When the solutions move away from the equilibrium solution then the equilibrium solution is called a 'source'. This is shown by arrows in directions away from the equilibrium solution. Conversely, when the solutions move towards the equilibrium solution then the equilibrium solution is called a 'sink'. This is shown by arrows in directions away from the equilibrium solution.

The phase portrait developed for Eq^n (11) give a very important information about nature of the equation and its solutions. It turns out that irrespective of the magnitude of 'a', the "fate" of the system remains unchanged as long as the sign of a remains the same. However, however small, if a change in magnitude of 'a' is made close to $a = 0$ such that the sign of 'a' changes then the fate of the system changes as $t \to \infty$. In such a case the system is said to have a bifurcation at $a = 0$. While analyzing a state-space model, it is important to identify bifurcations in the system as even small changes in the parameters about the bifurcation point can lead to drastically different temporal behaviour of the system.

Now we consider the case of a second order system which is described either by a second order ODE or by a set of two first order ODE's. As seen previously a second order ODE can be converted to a set of two first order ODE's. So let us consider a family of two first order ODE's.

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad - (12)$$

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} \qquad - (13)$$

It can be seen from the above equation that the vector $[0 \;\; 0]^T$ is always a solution of the system. Further, this solution is an equilibrium solution. To determine other equilibrium solutions, we need to determine the null space of $\underline{\underline{A}}$. We can deduce quickly the followings:

- If $\det(\underline{\underline{A}}) \neq 0$ then the only equilibrium solution is $[0 \;\; 0]^T$.
- If $\det(\underline{\underline{A}}) = 0$, then the null space is populated. Each vector in the null space is an equilibrium solution. Within a multiplicative constant coming from the field, all the points lying on the straight line joining $[0 \;\; 0]^T$ and the member of the null space are solutions. Such solutions are called straight line solutions.

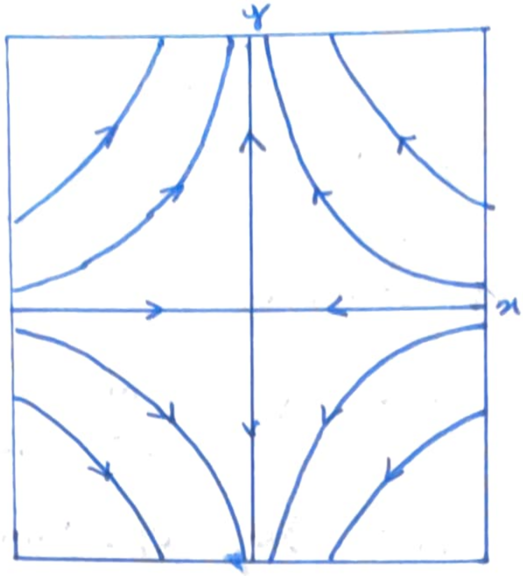
From the foregoing discussion, it can be seen that the nature of solutions can be commented upon by analysing $\underline{A}$. The analysis can be further extended to higher order systems as well but here we first consider 2x2 systems.

Consider a case where $\underline{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. It can be easily deduced that the eigenvalues are λ_1 and λ_2. The corresponding eigenvectors are $[1\ 0]^T$ and $[0\ 1]^T$, respectively. Therefore,

$$\underline{x} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \quad -(14)$$

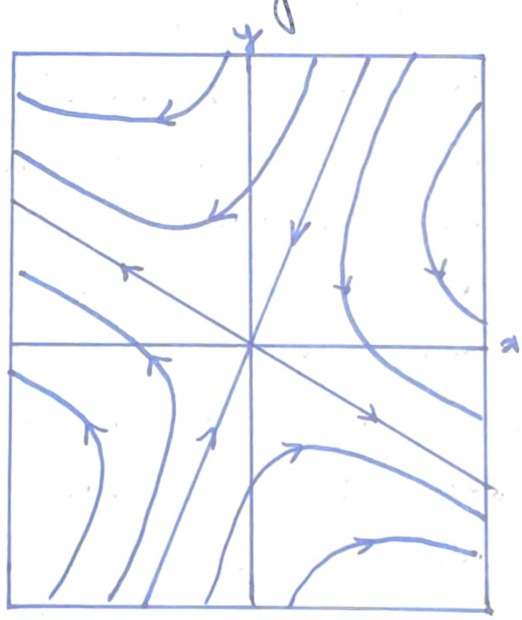
$[1\ 0]^T$ and $[0\ 1]^T$ are the straight line equilibrium solutions. They lie along the x- and the y-axis, respectively. The linear combination operation given by Eq^n (14) gives all possible solutions.

Consider a case where $\lambda_1 < 0 < \lambda_2$. Due to the exponential terms appearing in Eq^n (14), the first term in the RHS of Eq^n (14) will decay to zero and the second term will rise to infinity as $t \to \infty$. Therefore, the straight line solutions along $[1\ 0]^T$, i.e. x-axis, can be identified as "stable" and x-axis can be referred to as the stable line. Conversely, the y-axis can be referred to as the unstable line. This simply means that as $t \to \infty$, all the solutions tend towards the x-axis and move away from the y-axis. Accordingly, the phase portrait of the system can be drawn.



As mentioned before, the arrows in the above phase portrait show the direction of time. Such solutions are called saddle solutions.

One can have a system with $\underline{A}$ such that the condition of $\lambda_1 < 0 < \lambda_2$ is satisfied but the eigenvectors are not $[1\ 0]^T$ and $[0\ 1]^T$ i.e. the eigenvectors are not along the x-y axes. In such a case, the stable and unstable lines can be identified and a phase portrait can be drawn which would look something like the one shown below.



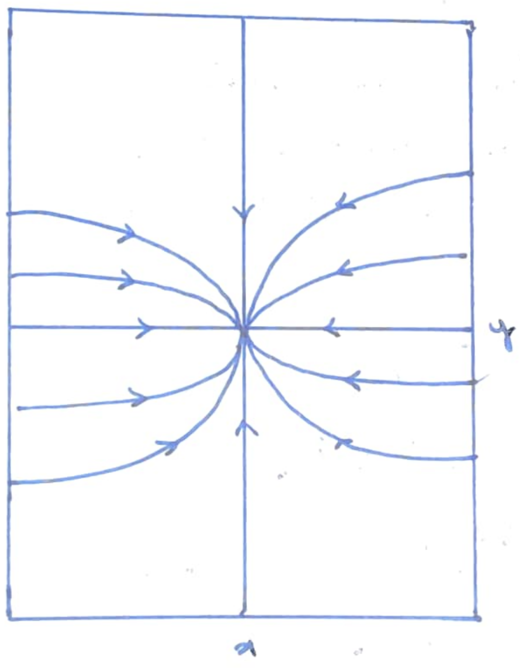
The previous analysis can be extended for two new cases as described here. With $\underline{A}$ of the form considered previously and $\lambda_1 < \lambda_2 < 0$, it can be seen that both the terms on the RHS of Eq^n (14) decay exponentially to zero as $t \to \infty$. Hence both the eigenvectors represent stable lines. Conversely for $\lambda_1 > \lambda_2 > 0$, both the lines represent unstable lines. The other solutions can be analyzed as follows:

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow x = c_1 e^{\lambda_1 t} \; ; \; y = c_2 e^{\lambda_2 t}$$

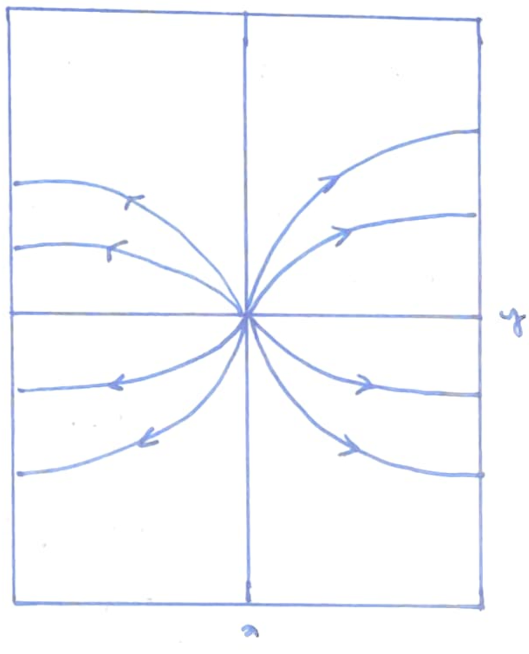
$$\Rightarrow \frac{dy}{dx} = \left(\frac{\lambda_2}{\lambda_1}\right)\left(\frac{c_2}{c_1}\right) \cdot e^{(\lambda_2 - \lambda_1) t}$$

Since $\lambda_2 > \lambda_1$, the slopes of x-y phase lines should tend to ∞ as $t \to \infty$. The converse conclusion can be drawn for the other case. Accordingly, one can draw "sink" and "source" solutions as shown below.



$\lambda_1 < \lambda_2 < 0$

sink



$\lambda_1 > \lambda_2 > 0$

source

The eigenvalues, λ_1 and λ_2, considered so far are real and distinct eigenvalues. Let us now consider a case in which the eigenvalues are complex numbers. Firstly we consider $\underline{\underline{A}} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$. The eigenvalues can be calculated to be $\lambda_1 = i\beta$ and $\lambda_2 = -i\beta$. The eigenvector corresponding to $\lambda = i\beta$ can be calculated to be $[1 \quad i]^T$. Hence, one of the possible solutions is

$$\underline{x}(t) = e^{i\beta t}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad -(15)$$

But $e^{i\beta t} = \cos\beta t + i\sin\beta t$. Therefore,

$$\underline{x}(t) = \begin{bmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{bmatrix}$$

$$\Rightarrow \quad \underline{x}(t) = \begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix} + i\begin{bmatrix} \sin\beta t \\ \cos\beta t \end{bmatrix}$$

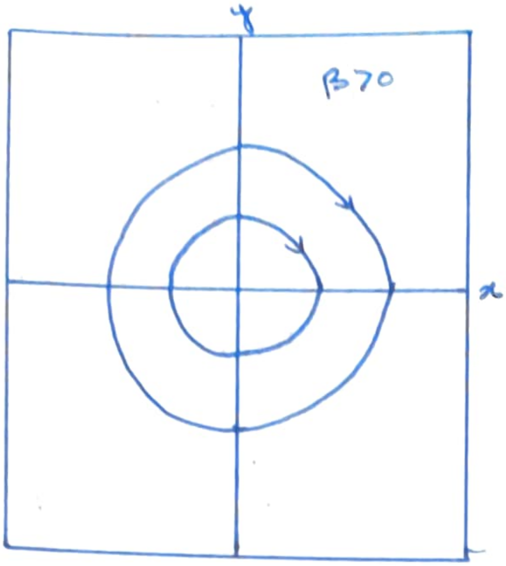
$$\Rightarrow \quad \underline{x}(t) = \underline{x}_{Re}(t) + i\,\underline{x}_{im}(t)$$

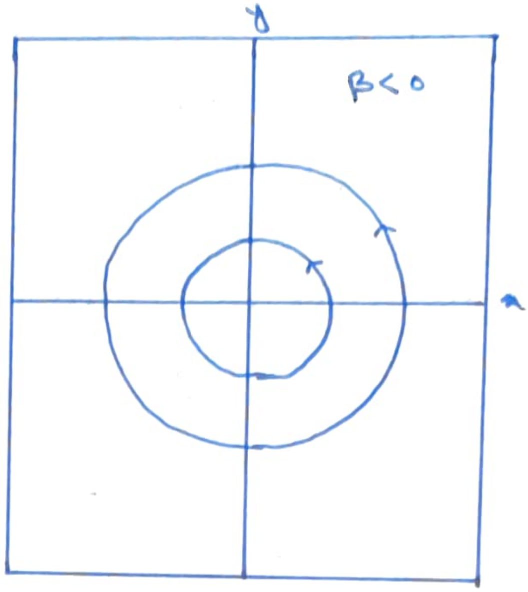
By substitution, it can be easily seen that $\underline{x}_{Re}(t)$ and $\underline{x}_{im}(t)$ are individually the solutions to the original problem $\underline{x}' = \underline{\underline{A}}\,\underline{x}$. Hence, using linearity, we can write the solution as

$$\underline{x}(t) = c_1\begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix} + c_2\begin{bmatrix} \sin\beta t \\ \cos\beta t \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = \begin{bmatrix} c_1\cos\beta t + c_2\sin\beta t \\ -c_1\sin\beta t + c_2\cos\beta t \end{bmatrix} \qquad -(16)$$

Each of the above functions are periodic. Hence, the solutions will also be periodic and they will be referred to as "centre" solutions.





The eigenvector considered in the previous case was $[1 \;\; i]^T$ corresponding to the eigenvalue of $\lambda = i\beta$. The entire procedure may be repeated to do a similar analysis for $\lambda = -i\beta$. Centre solutions will be obtained in this case also.

The eigenvalues considered in the previous case were purely imaginary. Now let us consider the matrix $\underline{A} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, $\alpha, \beta \neq 0$. The eigenvalues in this case are $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. The eigenvector corresponding to $\lambda = \alpha + i\beta$ is again $[1 \;\; i]^T$.

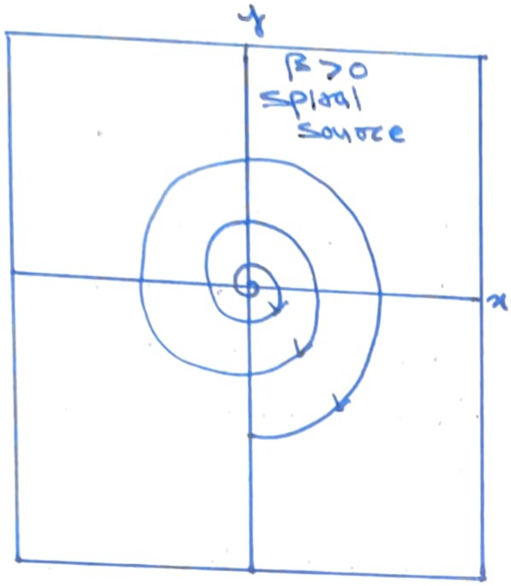
$$\Rightarrow \underline{x}(t) = e^{(\alpha+i\beta)t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

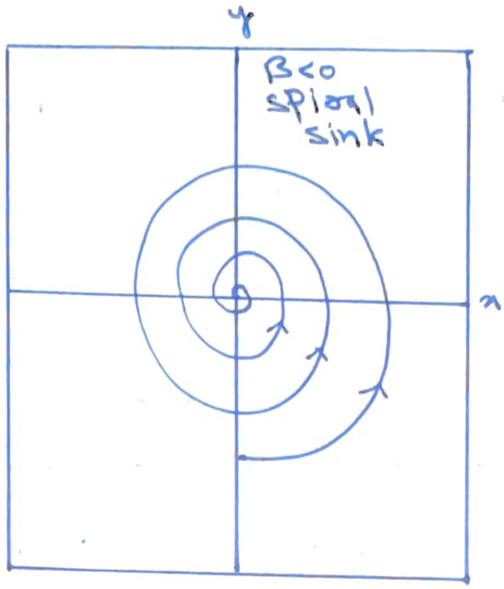
$$= e^{\alpha t} \begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix} + i e^{\alpha t} \begin{bmatrix} \sin\beta t \\ \cos\beta t \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = c_1 e^{\alpha t} \begin{bmatrix} \cos\beta t \\ -\sin\beta t \end{bmatrix} + c_2 e^{\alpha t} \begin{bmatrix} \sin\beta t \\ \cos\beta t \end{bmatrix} \quad -(17)$$

$$\Rightarrow \underline{x}(t) = \begin{bmatrix} e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t) \\ e^{\alpha t}(-c_1 \sin\beta t + c_2 \cos\beta t) \end{bmatrix} \quad -(18)$$

A comparison of Eq^n (16) and Eq^n (18) will quickly tell us that an extra term of $e^{\alpha t}$ appears in the latter case. Depending upon the sign of α, $e^{\alpha t}$ will grow or decay with time affecting the periodic part. Hence "spiral" solutions will be obtained which can be characterized as spiral source or spiral sink.





Lastly, we consider the case of repeating eigenvalus. Let $\underline{\underline{A}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$. The eigenvalues are all λ. For this case, any and every vector is an eigenvector. Hence,

$$\underline{x}(t) = \alpha e^{\lambda t} \underline{v}$$

Hence, each solution is a straight line solution passing through $[0 \;\; 0]^T$. Depending upon the sign of λ, the solutions either move away $(\lambda > 0)$ or move towards $[0 \;\; 0]^T$. A more interesting case is for $\underline{\underline{A}} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

In this case also, the eigenvalues are all λ. But the eigenvector is $[1 \;\; 0]^T$.

$$\Rightarrow \quad \underline{x}(t) = \alpha e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow \quad \frac{dx}{dt} = \lambda x + y$$

$$\frac{dy}{dt} = \lambda y$$

$$\Rightarrow \quad y = \beta e^{\lambda t}$$

$$\Rightarrow \quad \frac{dx}{dt} = \lambda x + \beta e^{\lambda t} \qquad - (19)$$

The solution of equation of the form appearing in Eq^n (19) is given by

$$x(t) = \alpha e^{\lambda t} + \mu t e^{\lambda t} \qquad - (20)$$

$$\Rightarrow \frac{dx}{dt} = \alpha\lambda e^{\lambda t} + \mu\lambda t e^{\lambda t} + \mu e^{\lambda t} \qquad - (21)$$

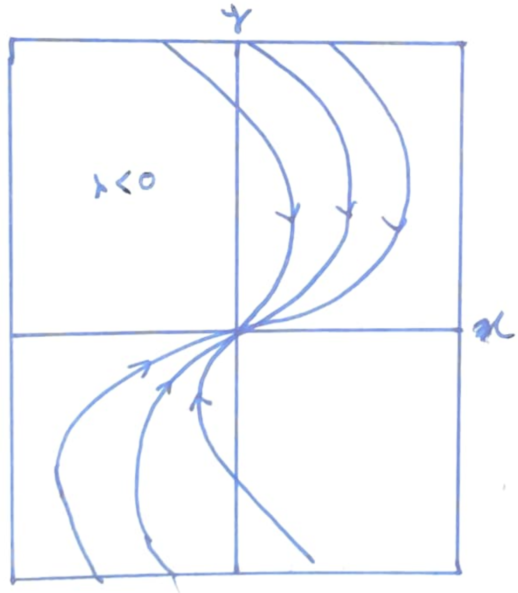
From Eq^n's (19) and (21), $\mu = \beta$.

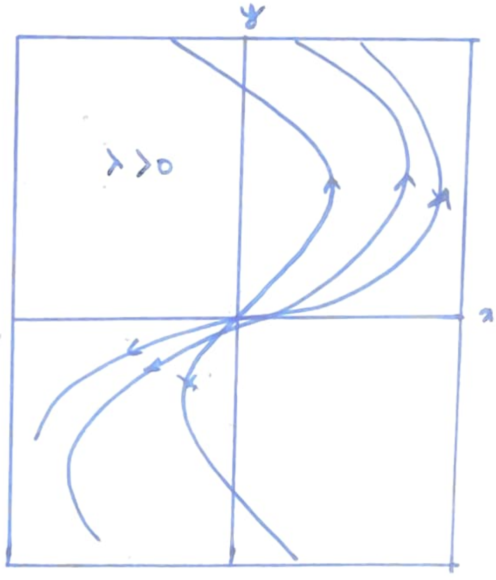
$$\Rightarrow \quad x(t) = \alpha e^{\lambda t} + \beta t e^{\lambda t}$$

$$y(t) = \beta e^{\lambda t}$$

$$\Rightarrow \quad \underline{x}(t) = \alpha e^{\lambda t}\begin{bmatrix}1\\0\end{bmatrix} + \beta e^{\lambda t}\begin{bmatrix}t\\1\end{bmatrix} \qquad - (22)$$

which gives the phase portrait as given below.





In the analysis carried out till now, the matrix $\underline{A}$ had one of the following forms:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Eigenvalues and eigenvectors can be quickly determined if $\underline{A}$ is in one of the above forms. A matrix in the above form is said to be in canonical form. Conversion of a matrix in any form to the canonical form can be carried out using "similarity transformation". We demonstrate it using the following example.

Consider a case when $\underline{A} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}$. Clearly, $\underline{A}$ is not in the canonical form. The eigenvalues and the corresponding eigenvectors are:

$$\lambda_1 = -1 \qquad \underline{v}_1 = [1 \quad 1]^T$$

$$\lambda_2 = -2 \qquad \underline{v}_2 = [0 \quad 1]^T$$

We can see that the solution corresponding to $\underline{A}\,\underline{X} = \underline{X}'$ will be a sink solution with stable lines as those passing through $[0 \quad 0]^T$ and $[1 \quad 1]^T$, and $[0 \quad 0]^T$ and $[0 \quad 1]^T$. Let us now consider the matrix $\underline{P} = [\underline{v}_1 \mid \underline{v}_2]$.

$$\underline{P} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{P}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\Rightarrow \underline{P}^{-1}\underline{A}\,\underline{P} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

The above matrix can be seen to be in the canonical form with eigenvalues as diagonal elements.

Since $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ is a matrix, it can be used to cast a problem

$$\underline{Y}' = (\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}})\underline{Y}$$

$$\Rightarrow \underline{\underline{P}}\underline{Y}' = [(\underline{\underline{A}}\,\underline{\underline{P}})\underline{Y}]$$

$$\Rightarrow \underline{\underline{P}}\underline{Y}' = \underline{\underline{A}}(\underline{\underline{P}}\underline{Y})$$

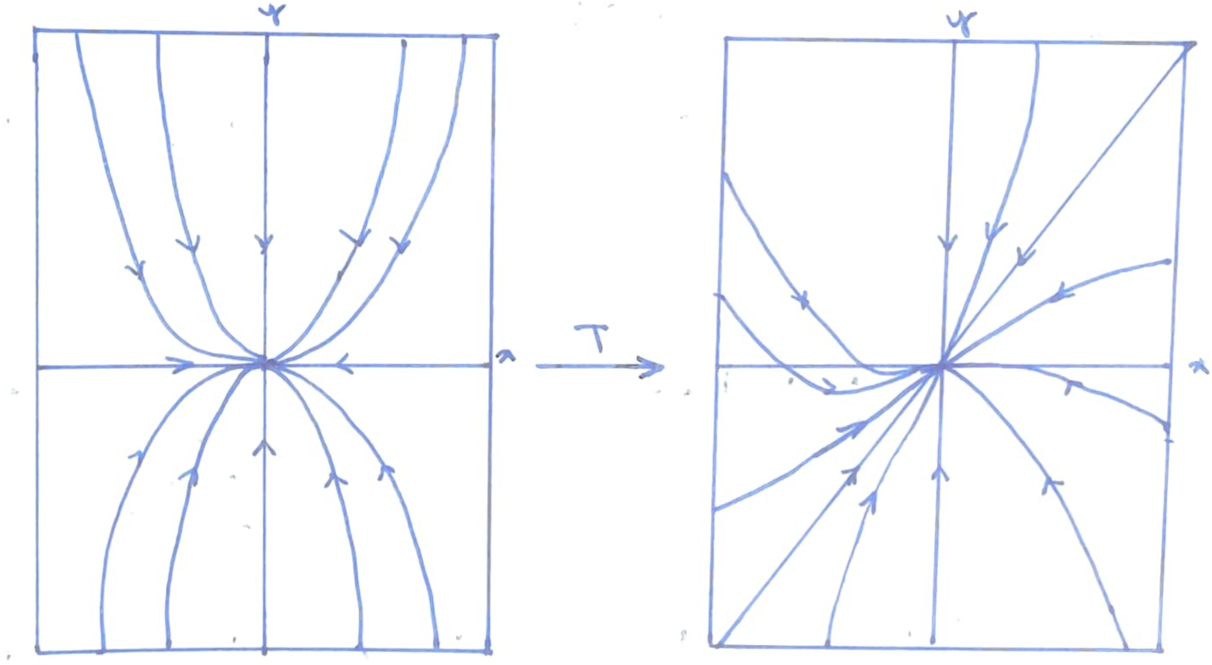
Hence $\underline{\underline{P}}\underline{Y} = \underline{X}$. But the matrix $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$ is in the canonical form. Hence

$$Y(t) = \alpha e^{-t}\begin{bmatrix}1\\0\end{bmatrix} + \beta e^{-2t}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\Rightarrow \underline{\underline{P}}\underline{Y} = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\left\{\alpha e^{-t}\begin{bmatrix}1\\0\end{bmatrix} + \beta e^{-2t}\begin{bmatrix}0\\1\end{bmatrix}\right.$$

$$\Rightarrow \underline{X} = \alpha e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + \beta e^{-2t}\begin{bmatrix}0\\1\end{bmatrix}$$

Hence, the similarity transformation has changed the features of the phase portrait keeping the nature of solutions (sink in this case) the same.



Now consider $\underline{A} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ which is not in the canonical form. $\lambda_1 = 2i$, $\lambda_2 = -2i$. $\underline{v}_1 = [1 \quad 2i]^T$.

Hence,

$$\underline{x}(t) = \alpha e^{2it} \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = \alpha(\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ 2i \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = c_1 \begin{bmatrix} \cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ 2\cos 2t \end{bmatrix}$$

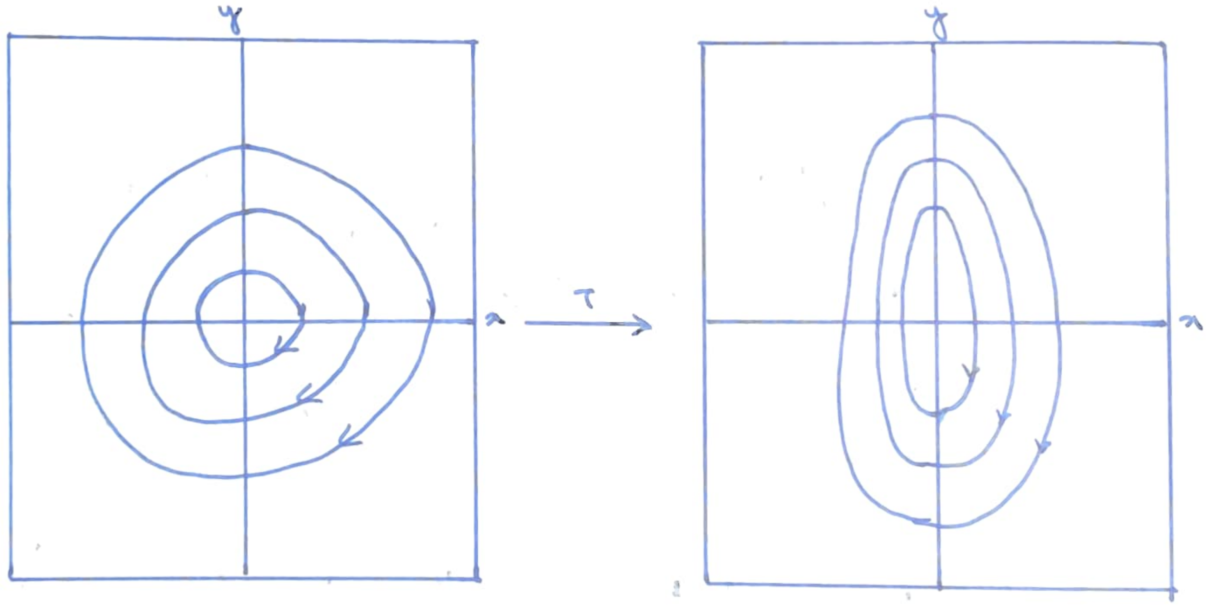
$$\Rightarrow \quad x(t) = c_1 \cos 2t + c_2 \sin 2t$$

The transformation matrix $\underline{\underline{P}}$ in this case is made from the augmentation of real and imaginary parts of $\underline{v}$. Since $\underline{v}_1 = \begin{bmatrix} 1 \\ 2i \end{bmatrix}$

$$\underline{\underline{P}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{P}}^{-1} \underline{\underline{A}}\, \underline{\underline{P}} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \quad \text{- canonical form.}$$

Hence, the transformation of the phase portrait can be seen as below:



The equations considered till now were homogeneous. All of them were of the form $\underline{x}' = \underline{\underline{A}}\,\underline{x}$.
We now consider the solution of non-homogeneous initial value problems of the form $\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b}(t)$ with an initial condition specified as $\underline{x}(0) = \underline{x}_0$.
Similarity transformation is helpful in solving such systems. We review some mathematical concepts below:

Similar matrices: If $\underline{\underline{P}}$ is any non-singular matrix such that

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{B}}$$

then $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are said to be similar matrices.
Consider

$$\underline{\underline{B}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$$

$$\Rightarrow \underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,(\underline{\underline{P}}\,\underline{\underline{P}}^{-1})$$

$$\Rightarrow \underline{\underline{B}}\,\underline{\underline{P}}^{-1} = \underline{\underline{P}}^{-1}\underline{\underline{A}}$$

$$\Rightarrow \underline{\underline{B}}\,\underline{\underline{P}}^{-1}\underline{x} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x}$$

If $\underline{x}$ is an eigenvector of $\underline{\underline{A}}$ then

$$\underline{\underline{B}}\,\underline{\underline{P}}^{-1}\underline{x} = \underline{\underline{P}}^{-1}\lambda\underline{x}$$

$$\Rightarrow \underline{\underline{B}}(\underline{\underline{P}}^{-1}\underline{x}) = \lambda(\underline{\underline{P}}^{-1}\underline{x})$$

If $\underline{\underline{P}}^{-1}\underline{x} = \underline{y}$ then

$$\underline{\underline{B}}\,\underline{y} = \lambda\underline{y}$$

The above equation means that if λ is an eigenvalue of $\underline{\underline{A}}$ with $\underline{x}$ as the eigenvector then λ will also be an eigenvalue of $\underline{\underline{B}}$ with eigenvector $P^{-1}\underline{x}$. Hence, similar matrices have same eigenvalues.

Diagonalization of matrices: Consider a matrix $\underline{\underline{P}}$ whose columns are made of eigenvectors of $\underline{\underline{A}}$.

$$\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{A}}[\underline{x}_1 | \underline{x}_2 | \dots | \underline{x}_n]$$

$$= [\underline{\underline{A}}\,\underline{x}_1 | \underline{\underline{A}}\,\underline{x}_2 | \dots | \underline{\underline{A}}\,\underline{x}_n]$$

$$= [\lambda_1 \underline{x}_1 | \lambda_2 \underline{x}_2 | \dots | \lambda_n \underline{x}_n]$$

$$= \underline{\underline{P}}\,\underline{\underline{\Lambda}}$$

where $\underline{\underline{\Lambda}}$ is the diagonal matrix

$$\underline{\underline{\Lambda}} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Hence an $n \times n$ matrix can be diagonalized if it has n linearly independent eigenvectors.

If an $n \times n$ matrix does not have n linearly independent eigenvectors then there exists a non-singular matrix $\underline{\underline{P}}$ such that

$$\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{J}}$$

where $\underline{\underline{J}}$ is called the Jordan matrix and has the following form.

$$\underline{\underline{J}} = \begin{bmatrix} \underline{\underline{J}}_1 & 0 & 0 & \dots & 0 \\ 0 & \underline{\underline{J}}_2 & 0 & \dots & 0 \\ 0 & 0 & \underline{\underline{J}}_3 & \dots & 0 \\ 0 & 0 & 0 & \dots & \underline{\underline{J}}_n \end{bmatrix}$$

where $\underline{\underline{J}}_i$'s are the Jordan blocks in which the eigenvalues are on the diagonal, 1's are on the first superdiagonal and the rest of the elements are all zeros.

As an example, a 3x3 Jordan block can be written as:

$$\underline{\underline{J}}_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{bmatrix}$$

If an nxn matrix has 'k' linearly independent eigenvectors then $\underline{\underline{P}}$ is constructed from these k eigenvectors. The remaining 'n-k' eigenvectors come from the generalized eigenvectors. For a 3x3 matrix, the following possibilities exist:

Only one eigenvector → one Jordan block

$$\underline{\underline{J}} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Two eigenvectors → Two Jordan blocks

$$\underline{\underline{J}} = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

or $$\underline{\underline{J}} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Three eigenvectors → Three Jordan blocks

$$\underline{\underline{J}} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Hence, the number of Jordan blocks equals the number of linearly independent eigenvectors of the matrix.

Now we look into the procedure for the determination of generalized eigenvectors. We take the case of a 3×3 matrix. Let us first consider the case of a system with only one eigenvector. The followings hold correct.

$$\underline{\underline{P}} = [\underline{x} \mid \underline{q}_1 \mid \underline{q}_2]$$

where $\underline{x}$ is the eigenvector and q_i's are the generalized eigenvectors.

$$\underline{\underline{A}}\,\underline{\underline{P}} = [\underline{\underline{A}}\,\underline{x} \mid \underline{\underline{A}}\,\underline{q}_1 \mid \underline{\underline{A}}\,\underline{q}_2]$$

$$\Rightarrow \underline{\underline{A}}\,\underline{\underline{P}} = [\lambda\underline{x} \mid \underline{\underline{A}}\,\underline{q}_1 \mid \underline{\underline{A}}\,\underline{q}_2]$$

$$\underline{\underline{P}}\,\underline{\underline{J}} = [\underline{x} \mid \underline{q}_1 \mid \underline{q}_2]\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= [\lambda\underline{x} \mid \underline{x}+\lambda\underline{q}_1 \mid \underline{q}_1+\lambda\underline{q}_2]$$

For $\underline{\underline{J}} = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}$, we equate $\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{J}}$.

$$\Rightarrow [\lambda\underline{x} \mid \underline{\underline{A}}\,\underline{q}_1 \mid \underline{\underline{A}}\,\underline{q}_2] = [\lambda\underline{x} \mid \underline{x}+\lambda\underline{q}_1 \mid \underline{q}_1+\lambda\underline{q}_2]$$

$$\Rightarrow \underline{\underline{A}}\,\underline{q}_1 = \underline{x}+\lambda\underline{q}_1 \text{ and } \underline{\underline{A}}\,\underline{q}_2 = \underline{q}_1+\lambda\underline{q}_2$$

$$\Rightarrow (\underline{\underline{A}}-\lambda\underline{\underline{I}})\underline{q}_1 = \underline{x} \text{ and } (\underline{\underline{A}}-\lambda\underline{\underline{I}})\underline{q}_2 = \underline{q}_1$$

Hence, $\underline{q}_1$ and $\underline{q}_2$ can be obtained as the solutions to the above two non-homogeneous problems. Let us now consider the case where two eigenvectors are available.

$$\underline{P} = [\underline{x}_1 \mid \underline{x}_2 \mid q]$$

$$\Rightarrow \underline{\underline{A}}\,\underline{\underline{P}} = [\underline{\underline{A}}\underline{x}_1 \mid \underline{\underline{A}}\underline{x}_2 \mid \underline{\underline{A}}q]$$

$$\Rightarrow \underline{\underline{A}}\,\underline{\underline{P}} = [\lambda\underline{x}_1 \mid \lambda\underline{x}_2 \mid \underline{\underline{A}}q]$$

$$\underline{\underline{P}}\,\underline{\underline{J}} = [\underline{x}_1 \mid \underline{x}_2 \mid q]\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$\Rightarrow \underline{\underline{P}}\,\underline{\underline{J}} = [\lambda\underline{x}_1 \mid \lambda\underline{x}_2 \mid \underline{x}_2 + \lambda\underline{q}]$$

$$\Rightarrow [\lambda\underline{x}_1 \mid \lambda\underline{x}_2 \mid \underline{\underline{A}}q] = [\lambda\underline{x}_1 \mid \lambda\underline{x}_2 \mid x_2 + \lambda q]$$

$$\Rightarrow (\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{q} = \underline{x}_2$$

By exchanging $\underline{x}_1$ and $\underline{x}_2$, another generalized vector can be obtained by solving

$$(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{q} = \underline{x}_1$$

and further by linearity, the following holds true.

$$(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{q} = \alpha\underline{x}_1 + \beta\underline{x}_2$$

Using the above concepts, we can now solve the inhomogeneous initial value problem. Given

$$\frac{d}{dt}\underline{x}(t) = \underline{\underline{A}}\,\underline{x}(t) + \underline{b}(t) \qquad -(23)$$

$$\Rightarrow \frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{x} + \underline{\underline{P}}^{-1}\underline{b}$$

$$\Rightarrow \frac{d}{dt}(\underline{\underline{P}}^{-1}\underline{x}) = \underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}}(\underline{\underline{P}}^{-1}\underline{x}) + \underline{\underline{P}}^{-1}\underline{b}$$

We identify $\underline{\underline{P}}^{-1}\underline{\underline{A}}\,\underline{\underline{P}} = \underline{\underline{\Lambda}}$ in the above equation. Denoting $\underline{\underline{P}}^{-1}\underline{x} = \underline{y}$ and $\underline{\underline{P}}^{-1}\underline{b} = \underline{g}$, we get

$$\frac{d\underline{y}}{dt} = \underline{\underline{\Lambda}}\,\underline{y} + \underline{g} \qquad -(24)$$

The above equation can be solved using the integrating factor method. The integrating factor is $e^{\underline{\underline{\Lambda}}t}$. Hence, we need to determine the exponential of a matrix. This is trivial for a diagonal matrix, as shown below.

$$e^{\underline{\underline{\Lambda}}t} = \underline{\underline{I}} + \underline{\underline{\Lambda}}t + \frac{(\underline{\underline{\Lambda}}t)^2}{2!} + \frac{(\underline{\underline{\Lambda}}t)^3}{3!} + \cdots$$

$$\Rightarrow e^{\underline{\underline{\Lambda}}t} = \begin{bmatrix} \sum_{n=0}^{\infty}\frac{(\lambda_1 t)^n}{n!} & 0 & 0 & \cdots & 0 \\ 0 & \sum_{n=0}^{\infty}\frac{(\lambda_2 t)^n}{n!} & 0 & \cdots & 0 \\ \\ 0 & 0 & 0 & \cdots & \sum_{n=0}^{\infty}\frac{(\lambda_n t)^n}{n!} \end{bmatrix}$$

$$\Rightarrow e^{\underline{\underline{\Lambda}}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} \qquad -(25)$$

Using the above result, one can solve for $\underline{y}(t)$ and the original problem of $\underline{x}(t)$ can be obtained from the relation $\underline{x}(t) = \underline{\underline{P}}\,\underline{y}(t)$.

The above solution procedure is valid if $\underline{\underline{A}}$ is diagonalizable. If this is not the case, then also the procedure remains the same except that now $\underline{\underline{J}}$ will appear in the equation instead of $\underline{\underline{\Lambda}}$.

We saw in the previous case the solution of a problem of the form $\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b}$. This can be expanded as follows:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1(t)$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2(t)$$

$$\vdots$$

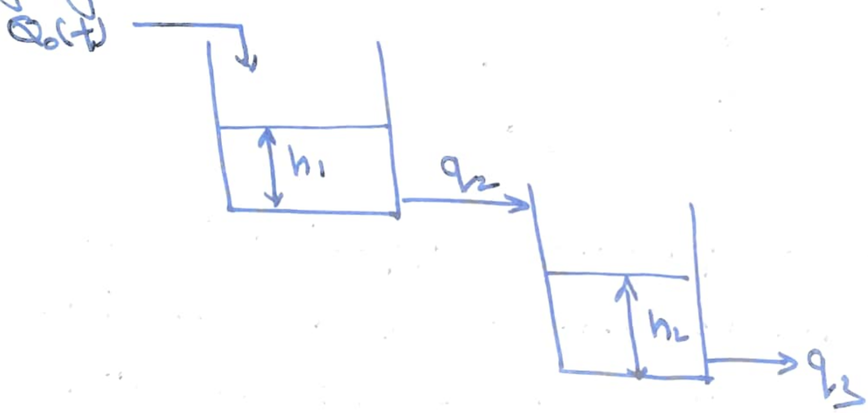
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n(t)$$

To see the example of this form, let us consider the two-tank system and the associated model equations (2) and (3).

$$A_1\frac{dh_1}{dt} = q_1 - q_2$$

$$A_2\frac{dh_2}{dt} = q_2 - q_3$$

Imagine a situation in which there is a time-varying input flow rate to tank 1, $Q_0(t)$.



The outlet lines are fitted with such valves as $q_2 = f_1(h_1)$ and $q_3 = f_2(h_2)$. In such a case, the model equations change to

$$\frac{dh_1}{dt} = \frac{1}{A_1} Q(t) - \frac{1}{A_1} f_1(h_1)$$

$$\frac{dh_2}{dt} = \frac{1}{A_2} f_1(h_1) - \frac{1}{A_2} f_2(h_2)$$

The above two set of equations can be recast as

$$\frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{A_1} f_1(h_1) & 0 \\ \frac{1}{A_2} f_1(h_1) & \frac{1}{A_2} f_2(h_2) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} Q_0(t) \\ 0 \end{bmatrix}$$

For linear valves, $f_1(h_1) = \alpha h_1$ and $f_2(h_2) = \beta h_2$.

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{A_1} & 0 \\ \frac{\alpha}{A_2} & \frac{\beta}{A_2} \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} Q_0(t) \\ 0 \end{bmatrix} \quad -(26)$$

which is of the form

$$\underline{x}' = \underline{\underline{A}}\,\underline{x} + \underline{b}$$

where $\underline{x} = [h_1 \quad h_2]^T$, $\underline{b} = \left[\frac{1}{A_1} Q_0(t) \quad 0\right]^T$ and

$$\underline{\underline{A}} = \begin{bmatrix} -\frac{\alpha}{A_1} & 0 \\ \frac{\alpha}{A_2} & \frac{\beta}{A_2} \end{bmatrix}.$$

Hence, given the parameters appearing in the matrix $\underline{\underline{A}}$ (valve coefficients, tank areas), and the time dependent input flowrate to tank 1, $Q_0(t)$, the evolution of liqurd levels in the two tanks can be determined using the aforementioned method.

The state-space models of the form $\underline{x}' = \underline{A}\underline{x} + \underline{b}$ can be further generalized as discussed below. Apart from the set of first order ordinary differential equations used to model the system, there can be associated algebraic equations governing the system. Further, the vector $\underline{b}$ itself can be a collection of equations. Hence, as the most generalized case, the following set of equations can be written.

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

The above set of equations represents a system which has "n" state variables. Hence, the state of the system is described by a $n \times 1$ vector $\underline{x} = [x_1 \; x_2 \; \ldots \; x_n]^T$. As can be seen from the previous example of two-tank liquid level system, a different state of the system is obtained for each $Q_o(t)$. Hence, $Q_o(t)$ is an "input variable". In the above set of equations, the system has "m" input variables. The vector $\underline{y}$ is said to be the output variable, and generally corresponds to the set of observables in the system. Hence, the generalized state-space equation for any system can be written as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

which can be further written as a set of matrix equations as

$$\begin{aligned} \dot{\underline{x}} &= \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{u} \\ \underline{y} &= \underline{\underline{C}}\,\underline{x} + \underline{\underline{D}}\,\underline{u} \end{aligned} \qquad -(27)$$

As seen from the previous examples, a thorough analysis of the matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$ and $\underline{\underline{D}}$ will provide insights into the temporal response and stability of the system under investigation.

Linearization of non-linear models

The cases that we considered till now all involved linear models for the description of dynamics of a process. However, a large number of instances involve non-linear dynamics. We will study non-linear dynamics in detail a little later. Before attempting to solve a non-linear problem, one may consider linearing the system and apply the methods described till now. We develop this technique of linearization here.

Let us first consider the simplest case of a system defined by a single ODE. In this case, let us assume one state variable and one input function.

$$\frac{dx}{dt} = f(x, u) \qquad -(28)$$

In case of the liquid level system under gravity driven flow condition, we have

$$\frac{dh}{dt} = \frac{F_{in}}{A} - \frac{\alpha\sqrt{h}}{A} \qquad -(29)$$

Hence, we have $x \equiv h$ and $u \equiv F_{in}$. The equation is non-linear due to $\sqrt{h}$.

For linearization, we make use of Taylor series expansion. If we know the steady state solution of the system, (x_s, u_s), then the Taylor series expansion of $f(x,u)$ about (x_s, u_s) can be written as:

$$f(x,u) = f(x_s,u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x-x_s) + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u-u_s)$$

$$+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_s,u_s}(x-x_s)^2 + \frac{1}{2}\left.\frac{\partial^2 f}{\partial u^2}\right|_{x_s,u_s}(u-u_s)^2$$

$$+ \frac{1}{2}\left.\frac{\partial^2 f}{\partial x \partial u}\right|_{x_s,u_s}(x-x_s)(u-u_s) + \ldots \quad -(30)$$

For linearization, we truncate the series after the first order derivatives.

$$\Rightarrow \frac{dx}{dt} = f(x,u) = f(x_s,u_s) + \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s}(x-x_s)$$

$$+ \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s}(u-u_s) \quad -(31)$$

Since $\frac{dx}{dt} = f(x,u)$, $\frac{dx_s}{dt} = f(x_s,u_s) = 0$.

Using this result and introducing the deviation variables $x^* = x - x_s$ and $u^* = u - u_s$, Eq^n (31) can be written as

$$\frac{dx^*}{dt} = \left.\frac{\partial f}{\partial x}\right|_{x_s,u_s} x^* + \left.\frac{\partial f}{\partial u}\right|_{x_s,u_s} u^* \quad -(32)$$

Using the notation $\left.\frac{\partial f}{\partial x}\right|_{x_s,u_s} = a$, $\left.\frac{\partial f}{\partial u}\right|_{x_s,u_s} = b$,

$$\frac{dx^*}{dt} = a x^* + b u^* \quad -(33)$$

From Eq^n (33) it can be seen that non-linear Eq^n (28) has been linearized. However, Eq^n (33) involves u^* which must be linearized, if the case be, to make the system linear. Further, the output function can also be linearized using the same technique. Imagine we have an output function $y = g(x, u)$ with the steady-state condition as $y_s = g(x_s, u_s)$. Like before,

$$g(x,u) = g(x_s, u_s) + \left.\frac{\partial g}{\partial x}\right|_{x_s, u_s} (x - x_s) + \left.\frac{\partial g}{\partial u}\right|_{x_s, u_s} (u - u_s)$$

$$\Rightarrow \quad y - y_s = c(x - x_s) + d(u - u_s)$$

$$\text{or} \quad y^* = c x^* + d u^* \qquad - (34)$$

A set of Eq^n's (33) and (34) present the linearized system. This philosophy can now be extended to a system described by multiple variables. We have

$$\frac{dx_1}{dt} = f_1(x_1, x_2 \ldots x_n, u_1, u_2, \ldots u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \ldots x_n, u_1, u_2 \ldots u_m)$$

$$y_1 = g_1(x_1, x_2 \ldots x_n, u_1, u_2, \ldots u_m)$$

$$\vdots$$

$$y_r = g_r(x_1, x_2 \ldots x_n, u_1, u_2, \ldots u_m)$$

Let the steady state be described by the set $(x_{1s}, x_{2s}, \ldots x_{ns}, u_{1s}, u_{2s}, \ldots u_{ms})$. Each of the above functions f_i's and g_i's can be expanded about the steady state using Taylor series expansion which is shown as an example as below.

$$f_i(x_1, x_2, \dots x_n, u_1, u_2, \dots u_m) = f_i(x_{1s}, x_{2s} \dots, x_{ns}, u_{1s}, u_{2s} \dots u_{ms})$$
$$+ \frac{\partial f_i}{\partial x_1}\Big|_{ss} (x_1 - x_{1s}) + \frac{\partial f_i}{\partial x_2}\Big|_{ss} (x_2 - x_{2s}) + \dots$$
$$+ \frac{\partial f_i}{\partial u_1}\Big|_{ss} (u_1 - u_{1s}) + \frac{\partial f_i}{\partial u_2}\Big|_{ss} (u_2 - u_{2s}) + \dots$$

$$g_i(x_1, x_2 \dots x_n, u_1, u_2 \dots u_m) = g_i(x_{1s}, x_{2s} \dots x_{ns}, u_{1s}, u_{2s} \dots u_{ms})$$
$$+ \frac{\partial g_i}{\partial x_1}\Big|_{ss} (x_1 - x_{1s}) + \frac{\partial g_i}{\partial x_2}\Big|_{ss} (x_2 - x_{2s}) + \dots$$
$$+ \frac{\partial g_i}{\partial u_1}\Big|_{ss} (u_1 - u_{1s}) + \frac{\partial g_i}{\partial u_2}\Big|_{ss} (u_2 - u_{2s}) + \dots$$

ss in the above equations represent steady state conditions. Introducing the deviation variables as $x_i - x_{is} = x_i^*$ and $u_i - u_{is} = u_i^*$ the above two equations can be represented as matrix equations as given below.

$$\frac{d\underline{x}^*}{dt} = \underline{\underline{A}}\,\underline{x}^* + \underline{\underline{B}}\,\underline{u}^* \qquad -(35)$$

$$\underline{y}^* = \underline{\underline{C}}\,\underline{x}^* + \underline{\underline{D}}\,u^* \qquad -(36)$$

where $\underline{x}^* = [x_1^* \quad x_2^* \quad \dots \quad x_n^*]^T$

$\underline{u}^* = [u_1^* \quad u_2^* \quad \dots \quad u_m^*]^T$

$\underline{y}^* = [y_1^* \quad y_2^* \quad \dots \quad y_r^*]^T$

$\underline{\underline{A}}_{ij} = \partial f_i / \partial x_j \,|_{ss}$

$\underline{\underline{B}}_{ij} = \partial f_i / \partial u_j \,|_{ss}$

$\underline{\underline{C}}_{ij} = \partial g_i / \partial x_j \,|_{ss}$

$\underline{\underline{D}}_{ij} = \partial g_i / \partial u_j \,|_{ss}$

Non-linear dynamics of process systems

The systems we considered till now were described by a set of linear ODE's. We will now consider the cases where the model equations are non-linear. We first consider the simplest case of a single first order ODE. Consider a system with a biological species and it is desired to describe the evolution of population of the species. The simplest model to describe the system will be the following ODE.

$$\frac{dx}{dt} = ax \qquad -(1)$$

This equation is the same as Eq^n (10) of the previous section. From the phase portrait developed previously, we can see that $x \to \infty$ as $t \to \infty$ when $a > 0$. This is clearly an unrealistic result. A more realistic model should incorporate the followings:

- the growth rate should be directly proportional to the population only at small populations.
- the growth rate should become negative when the population becomes large so as to "control" the population.

The model incorporating the above two conditions gives rise to a non-linear ODE known as the logistic population growth model. The corresponding equation is

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right) \qquad -(2)$$

'a' and 'N' in Eqⁿ (2) are positive parameters giving the growth rate at small populations and the "carrying capacity" of the system, respectively. We analyze the dynamics of a system with N=1.

$$\frac{dx}{dt} = ax(1-x)$$

$$\Rightarrow \int \frac{dx}{x(1-x)} = \int a\,dt$$

$$\Rightarrow \int \left(\frac{1}{x} - \frac{1}{1-x}\right) dx = \int a\,dt$$

$$\Rightarrow \quad x(t) = \frac{ke^{at}}{1+ke^{at}} \qquad -(3)$$

If the initial population of the system is x_0 then

$$x_0 = \frac{k}{1+k}$$

$$\Rightarrow k = \frac{x_0}{1-x_0}$$

$$\Rightarrow x(t) = \frac{x_0 e^{at}}{1-x_0+x_0 e^{at}} \qquad -(4)$$

We see the denominator $1-x_0+x_0e^{at}$ which makes the solution for logistic model different from the previous case.

Now we analyze the equilibrium solutions of the system. When $x_0 = 0$ in Eq^n (4), $x(t) = 0 \; \forall t$. Similarly when $x_0 = 1$, $x(t) = 0 \; \forall t$. Hence the equation has two equilibrium solutions as $x = 0$ and $x = 1$. However this result can be reached without explicitly solving the equation. For steady state we have $dx/dt = 0$

$$\Rightarrow \frac{dx}{dt} = ax(1-x) = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x = 1$$

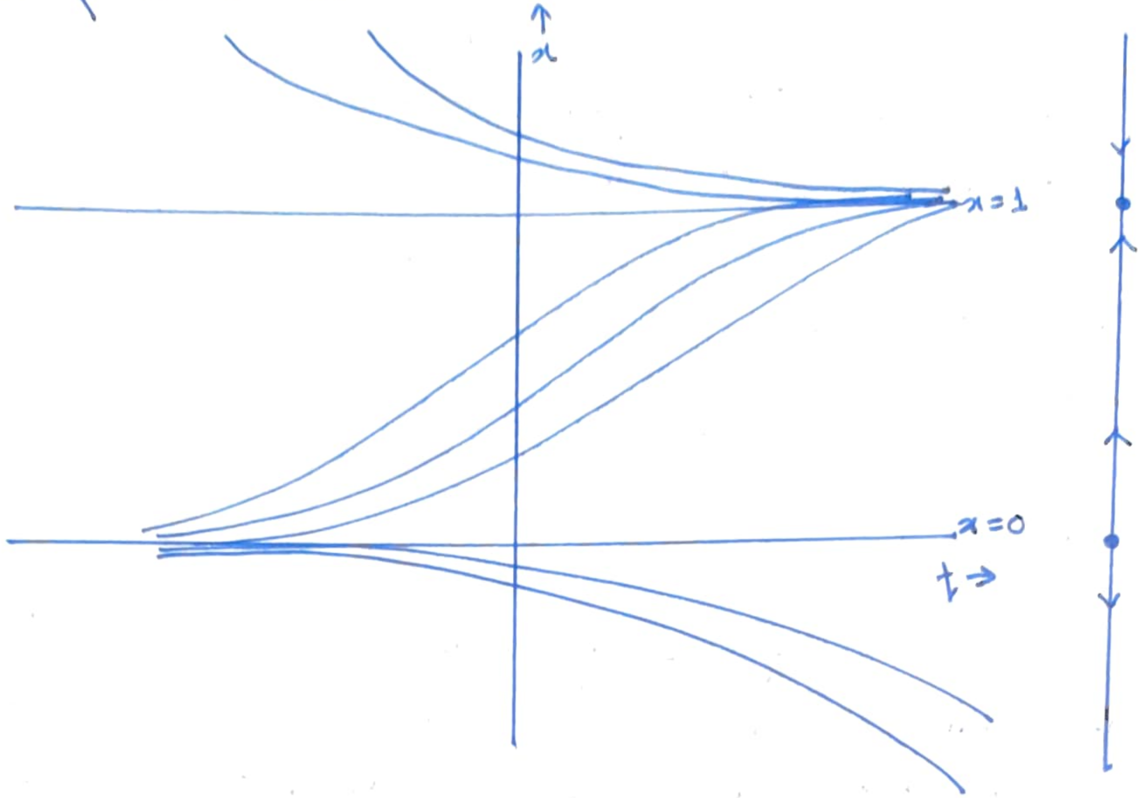
To draw the phase portrait of the system, we may analyze Eq^n (4) and sketch all possible solutions. However, this can be done also by analyzing the original ODE. The two equilibrium solutions are $x = 0$ and $x = 1$. Therefore, on a t-x plane, $x = 0$ and $x = 1$ lines are the solutions. We already have $x = 0$ and $x = 1$ as phase lines. We need to identify other non-equilibrium solutions. For this we divide the t-x plane into three regions:

$$x < 0 : \quad \frac{dx}{dt} = ax(1-x) < 0$$

$$0 < x < 1 : \quad \frac{dx}{dt} > 0$$

$$x > 1 : \quad \frac{dx}{dt} < 0$$

Using the slopes thus obtained, we can draw the phase portrait as shown below:



As before, we need to decide whether the equilibrium solution is a sink or a source. For this, we do the "derivative" test. We have

$$\frac{dx}{dt} = ax(1-x) = f_a(x)$$

$$\Rightarrow \frac{df_a(x)}{dx} = a - 2ax$$

At $x=0$, $\dfrac{df_a(x)}{dt} = a > 0$

This concludes that $x=0$ is a source, signified by arrows moving away from it. For $x=1$

$$\frac{df_a(x)}{dt} = -a < 0$$

Hence $x=1$ is a sink.

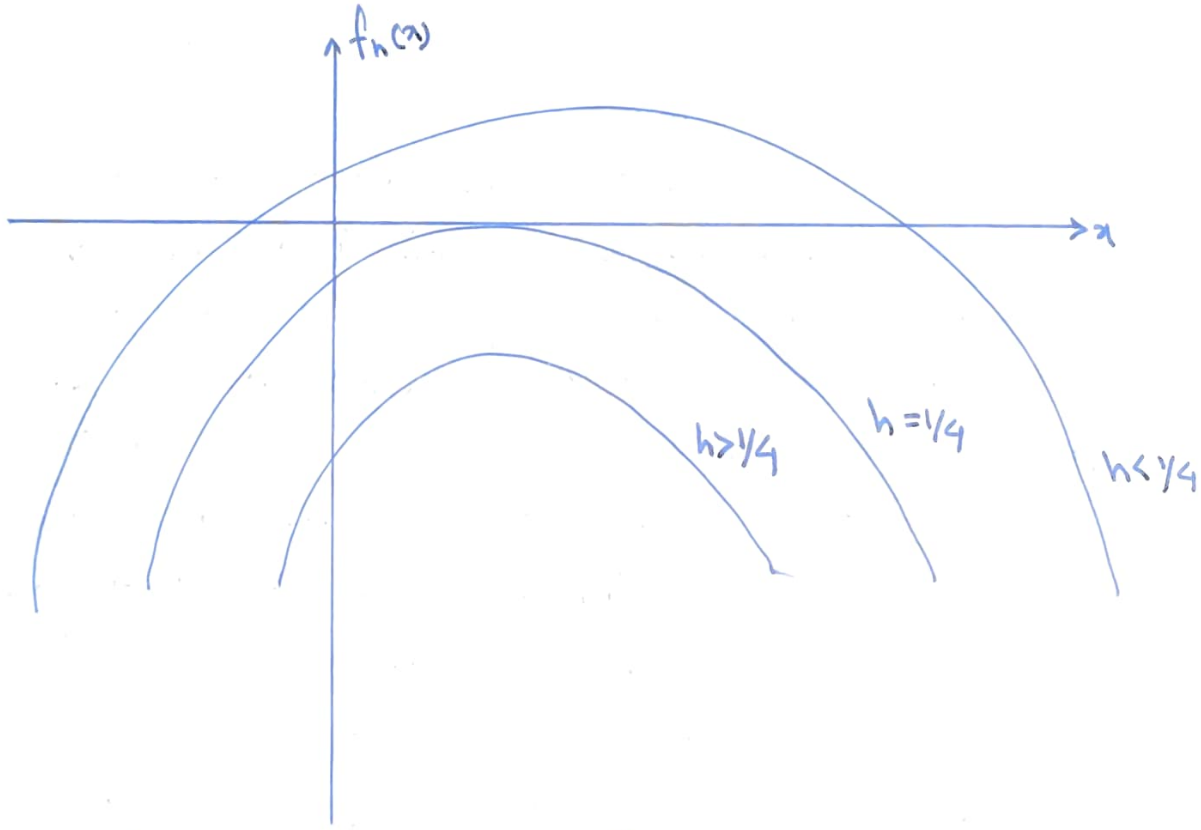
Now we consider a population described by logistic equation which is "harvested" at a constant rate 'h'. The governing ODE is (for $a = 1$)

$$\frac{dx}{dt} = x(1-x) - h \qquad -(5)$$

'h' is a positive parameter in Eq^n (5). It is desired to analyze the dynamics of the system as a function of h. Defining

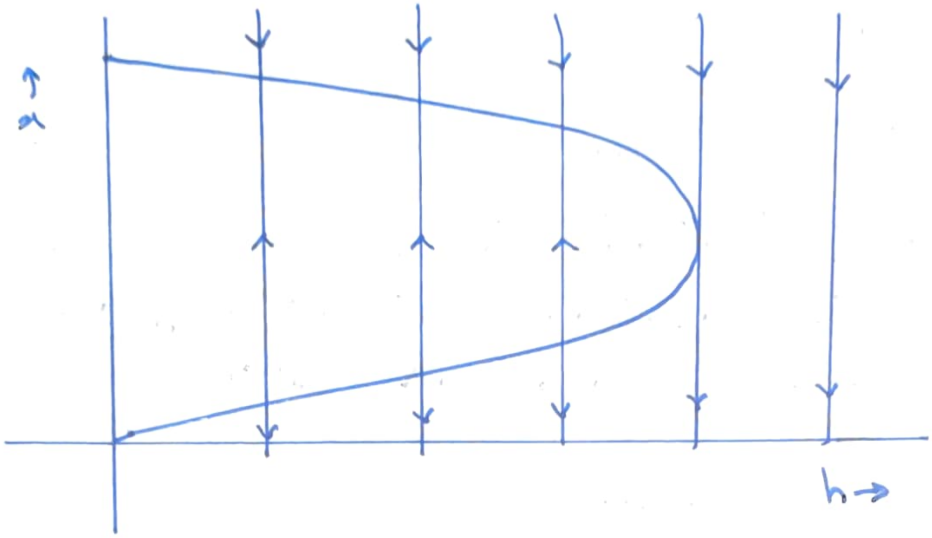
$$f_h(x) = x(1-x) - h \qquad -(6)$$

we identify bifurcation in the system as h changes. Eq^n (6) shows bifurcation at $h = 1/4$. A sketch of $f_h(x)$ as a function of h is given below.



The equation has two roots for $0 < h < 1/4$, one root for $h = 1/4$ and no root for $h > 1/4$, hence the bifurcation at $h = 1/4$.

Since roots of $f_h(x)$ are obtained by setting $f_h(x) = 0 = dx/dt$, the roots of $f_h(x)$ are actually the equilibrium solutions of the system. Hence for $h < 1/4$, there are two equilibrium solutions, exactly one for $h = 1/4$ and none for $h > 1/4$. Clearly, the population vanishes for $h > 1/4$. Hence, the bifurcation diagram for the system can be drawn as shown below.



Analysis of the above bifurcation diagram is important. When there are two equilibrium solutions, then it can be seen that the population will persist only if the initial population is sufficiently large. Two equilibria merge as harvest rate increases. If the initial population is small then the population tends to diminish. Hence, we can classify the upper and lower portion of the parabola into source and sink solutions. The larger values are sink solutions while the lower values are the source solutions. Clearly, the fate of the population depends upon the harvest rate.

The systems considered till now were modelled using continuous functions. However, one often comes across situations where the time-dependent variables are not continuous but discrete. Population, for example, is an example of the variable which is discrete and therefore, it is appropriate to look for discrete systems. We analyze such systems now.

From Eq^n (1), we modelled the population of a species using an ODE which made the system continuous.

$$\frac{dx}{dt} = ax$$

Now we wish to find the discrete analogue of the above equation. The above equation literally means that the time rate of change of population depends the instantaneous population. So if the time is discretized then the population at time instance n+1 can be written in terms of population at time n. Mathematically,

$$x_{n+1} = a' x_n \qquad - (7)$$

where x_{n+1} and x_n are the populations at time instances n+1 and n, respectively. When 'a'' is positive then the population keeps on increasing and when 'a'' is negative, the populations will keep on decreasing. Of course the case of $a'<0$ is physically not true in case of population dynamics. Eq^n (7) as the discretize form of Eq^n (1) can be obtained as follows:

$$\frac{dx}{dt} = ax$$

$$\Rightarrow \frac{x_{n+1} - x_n}{\Delta t} = a x_n$$

$$\Rightarrow x_{n+1} = a \Delta t x_n + x_n$$

$$\Rightarrow x_{n+1} = (a \Delta t + 1) x_n$$

$$\Rightarrow x_{n+1} = a' x_n$$

where $a' = 1 + a \Delta t$

It was seen earlier that a more appropriate model for population dynamics is the logistic equation given by Eq^n (2).

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right)$$

$$\Rightarrow \frac{x_{n+1} - x_n}{\Delta t} = a x_n \left(1 - \frac{x_n}{N}\right)$$

$$\Rightarrow x_{n+1} = x_n + a \Delta t x_n \left(1 - \frac{x_n}{N}\right)$$

$$\Rightarrow x_{n+1} = (a \Delta t + 1) x_n \left(1 - \frac{x_n}{\frac{(a\Delta t + 1)}{a \Delta t} N}\right)$$

$$\Rightarrow x_{n+1} = a' x_n \left(1 - \frac{x_n}{N'}\right) \qquad - (8)$$

where $a' = 1 + a \Delta t$

$$N' = N\left(\frac{1 + a\Delta t}{a \Delta t}\right)$$

A generic discretized and normalized logistic equation can be written as

$$x_{n+1} = \lambda x_n (1 - x_n) \qquad - (9)$$

To determine the evolution of population following the continuous logistic equation, one requires an initial population, x_0. Similarly, with x_0 known for $n=0$ in the discrete logistic equation (2), one can determine the evolution of the population by iterating the quadratic function $f_\lambda = \lambda x(1-x)$. This gives rise to what is referred to as the "logistic map".

It was seen from the continuous logistic model that there exist equilibrium solutions of the system. Hence, the discrete model should also be in a position to capture the same dynamics. In order to appreciate that, we need to invoke the idea of "fixed points". Evolution of population following Eq^n (7) can be obtained by iterating Eq^n (7) with a "seed" x_0 resulting in the "orbit" of x_0 as the following sequence.

$$f(x) = a'x$$

$x_0,\ f(x_0) = x_1,\ f(x_1) = x_2$ . . . . and so on.

For a function, there may exist a point x_0 such that $f(x_0) = x_0$. This will result in the orbit of x_0 as a constant sequence

$x_0,\ f(x_0) = x_0,\ f(x_0) = x_0$ . . . .

Hence the future of the system remains unchanged on attaining x_0 and such a point is called the fixed point.

There may occur a situation in which the sequence may repeat after some fixed interval i.e. $f^n(x_0) = x_0$, resulting in a sequence

$x_0, x_1, \ldots x_{n-1}, x_0, x_1, \ldots x_{n-1}, x_0, x_1 \ldots x_{n-1}, x_0 \ldots$

Such periodic orbits of period 'n' are called n-cycles. One may check the following cases:

$f(x) = x^3$; fixed points $= 0, +1, -1$

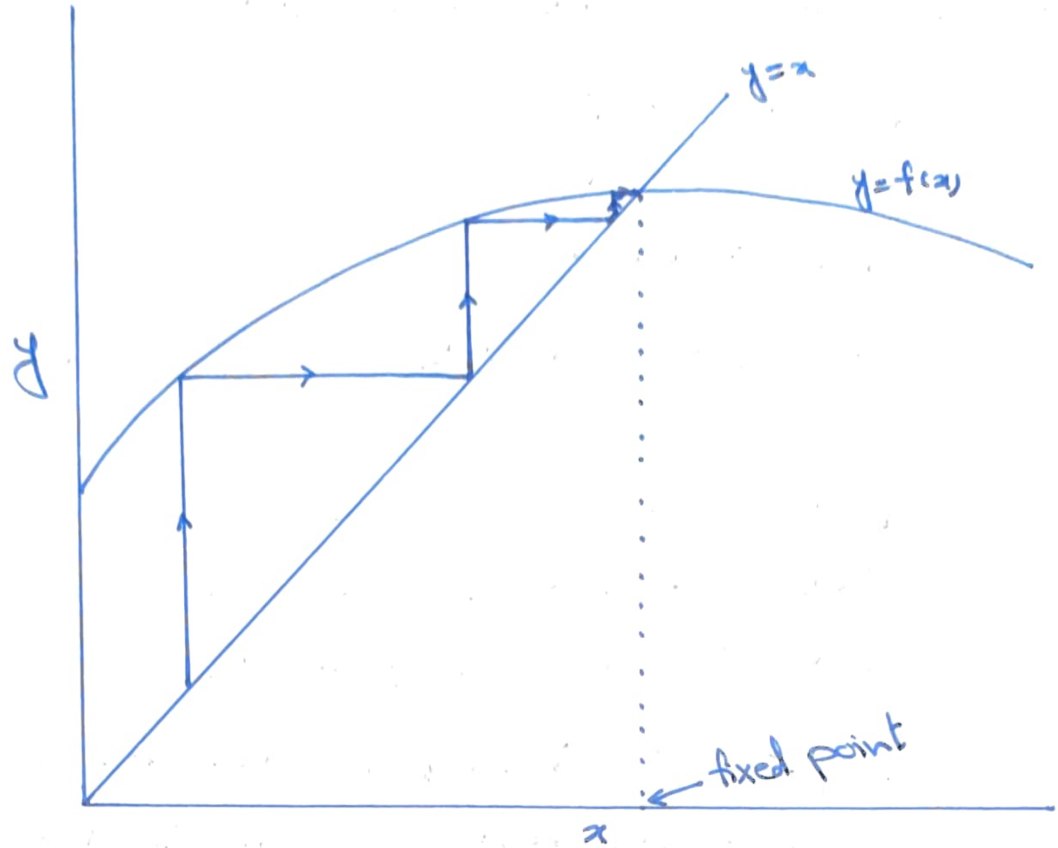
$f(x) = -x^3$; fixed point $= 0$

periodic points of period 2 at $x = \pm 1$

$$f(x) = \frac{(2-x)(3x+1)}{2} \quad ; \text{ 3-cycle}$$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2;\ x_3 = 0, \ldots$

"Graphical iteration" is a good way to analyze the orbits and obtain fixed points of one dimensional discrete dynamical systems. To analyze $y = f(x)$, it is required to draw the two curves $y = f(x)$ and $y = x$ simultaneously. It can be seen that the point of intersection is a fixed point. Fixed point and n-cycles are shown in the following figures illustrating the graphical iteration technique.



Like the equilibrium solutions of an ODE, the nature of fixed points can also be categorized as source or sink. The following are the criteria:

If $f(x)$ has a fixed point at x_0 then

- x_0 is a sink if $|f'(x_0)| < 1$
- x_0 is a source if $|f'(x_0)| > 1$
- x_0 gives no information about source or sink if $f'(x_0) = \pm 1$.

Some other terms for source and sink are repelling and attracting fixed point. If the fixed point is neither repelling nor attracting then it is called indifferent or neutral.

Consider three functions given below:

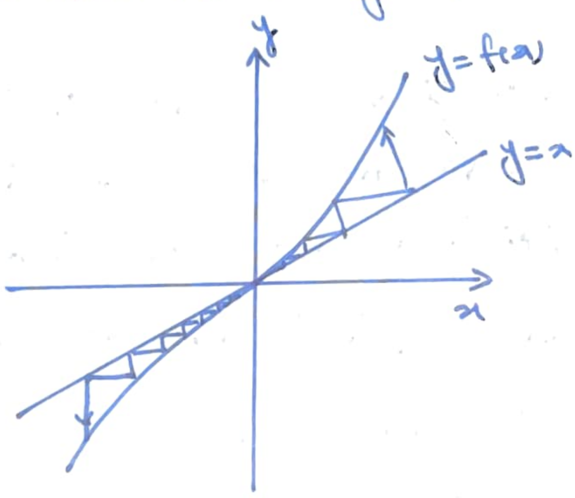
$$f(x) = x + x^3$$
$$g(x) = x - x^3$$
$$h(x) = x + x^2$$

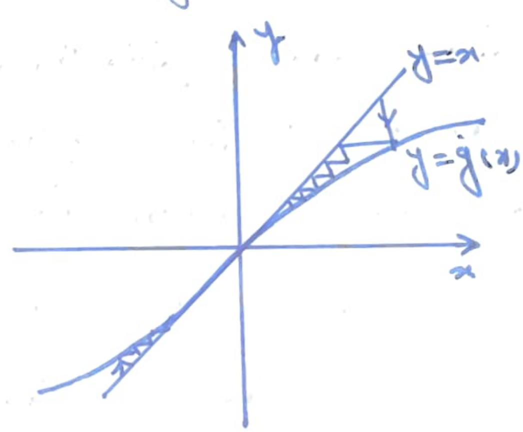
For all the three functions, $x_0 = 0$ is a fixed point.

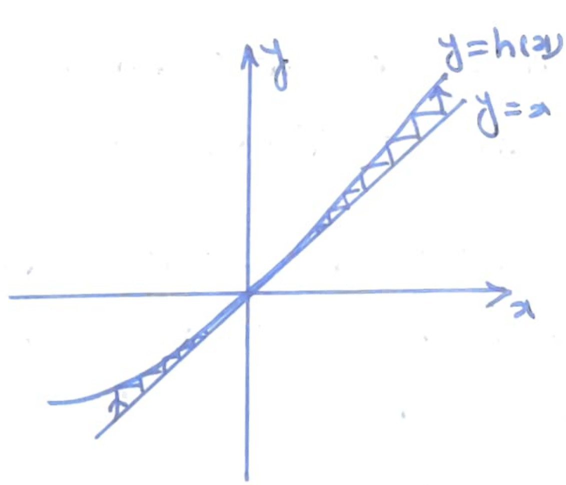
Further,

$$f'(x) = 1 + 3x^2 \Rightarrow f'(x_0) = 1$$
$$g'(x) = 1 - 3x^2 \Rightarrow g'(x_0) = 1$$
$$h'(x) = 1 + 2x \Rightarrow h'(x_0) = 1$$

Hence, as per our criterion, it may not be possible to comment upon the nature of the fixed point. This can be understood by the web diagrams shown below







Like continuous systems, we now analyze bifurcations in discrete systems. Consider the function with 'c' as a parameter.

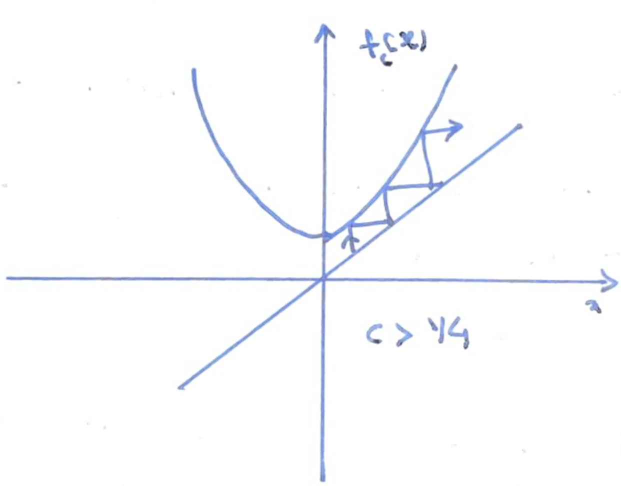
$$f_c(x) = x^2 + c$$

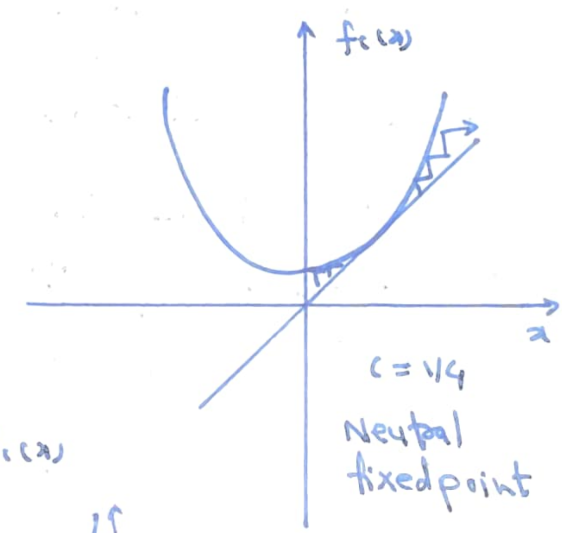
The fixed points for the above function can be obtained as the solutions of

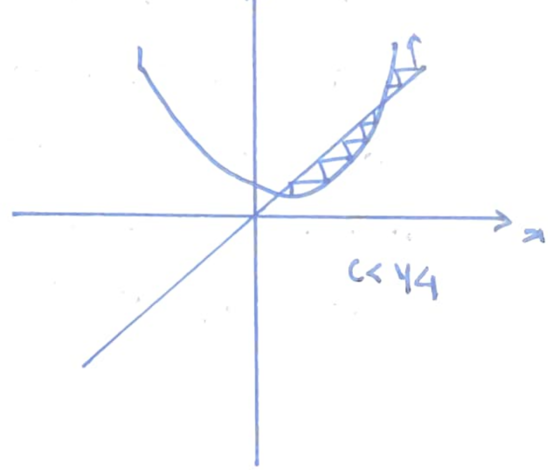
$$x^2 - x + c = 0$$

$$\Rightarrow x_0 = \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2}$$

Hence if $c > 1/4$, the system has no fixed points. It has a single fixed point if $c = 1/4$ and a pair of fixed points for $c < 1/4$. This is an example of "saddle-node" or tangent bifurcation. The situation can be graphically depicted as follows.







Let us now consider the discrete logistic equation.

$$f_\lambda(x) = \lambda x(1-x)$$

$$f'_\lambda(x) = \lambda - 2\lambda x$$

On solving $f_\lambda(x) = x$, we see that $x_0 = 0$ is a fixed point. $f'_\lambda(0) = \lambda$ for all λ. Hence, we have a bifurcation possible at $\lambda = 1$. The second fixed point is $x_0 = (\lambda-1)/\lambda$. Bounds on λ can be found out for possible bifurcations. When $\lambda = 1$, the two fixed points merge. The following behaviours can be deduced easily.

- $x_0 = 0$ is a repelling fixed point
- $x_0 = \frac{\lambda-1}{\lambda}$ is attracting for $\lambda > 1$
- $x_0 = \frac{\lambda-1}{\lambda}$ is repelling for $\lambda < 1$

This type of bifurcation is called exchange bifurcation.

Now we discuss in detail the logistic map which we mentioned previously. We have

$$f_\lambda(x) = \lambda x(1-x)$$

$\lambda = 0$ is a fixed point, as seen before. Since $f'_\lambda(x) = \lambda - 2\lambda x$, $f'_\lambda(0) = \lambda$. Hence, $x_0 = 0$ is attracting for $0 < \lambda < 1$. The same point becomes repelling for $\lambda > 1$. The second fixed point is $x_0 = (\lambda-1)/\lambda$. Hence $f'_\lambda(x_0) = 2-\lambda$.

Hence, the fixed point is attracting for $1 < \lambda \leq 3$ and repelling for $\lambda > 3$. For $\lambda = 3$, bifurcation called "period doubling" bifurcation occurs. This can be understood by the previous example.

$$f_c(x) = x^2 + c$$

$$x_0 = \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2}$$

$$f_c'(x) = 2x$$

$$\Rightarrow f_c'(x_0) = 1 \pm \sqrt{1-4c}$$

We see a bifurcation, called period doubling bifurcation at $c = -3/4$. The fixed point is attracting for $c \geq -3/4$ and it is repelling for $c < -3/4$. On application of the same analysis to the discrete logistic equation, we get period doubling bifurcation at $\lambda = 3$. A characteristic of period doubling bifurcation is the birth of cycles. In the discrete logistic equation, this happens for λ between 3 and 3.4 and periodic points are observed. However, for λ between 3.55 and 3.8, complete randomness in the values of $f(x)$ are observed and the system is said to be chaotic. This means that even a slight change in the initial condition of the system results in a totally different fate of the system. This fact was previously used for the generation of random numbers.

We revert back to continuous systems now which are modelled by a collection of ODE's which are non-linear. Consider the following example.

$$\frac{dx}{dt} = x + y^2 \qquad -(1)$$

$$\frac{dy}{dt} = -y \qquad -(2)$$

The equilibrium solution for the above system of equations can be obtained by the usual method of setting the derivatives to zero which yields the equilibrium solution as $(x_0, y_0) = (0, 0)$. When y is close to y_0, i.e. $y - y_0 \sim \epsilon$, $y^2 \to 0$. Hence, the equations can be linearized as follows:

$$\frac{dx}{dt} = x \qquad -(3)$$

$$\frac{dy}{dt} = -y \qquad -(4)$$

However, (3) and (4) will be valid only in the proximity of equilibrium solutions. We now explicitly solve (1) and (2). From Eq^n (2),

$$y = y_0 e^{-t}$$

$$\Rightarrow \frac{dx}{dt} = x + y_0^2 e^{-2t} \qquad -(5)$$

Eq^n (5) can be solved easily by IF method to give

$$x = \left(x_0 + \frac{1}{3} y_0^2\right) e^{t} - \frac{1}{3} y_0^2 e^{-2t} \qquad -(6)$$

From this detailed solution also, we find the equilibrium solutions as $(x_0, y_0) = (0, 0)$.

For the linearized case

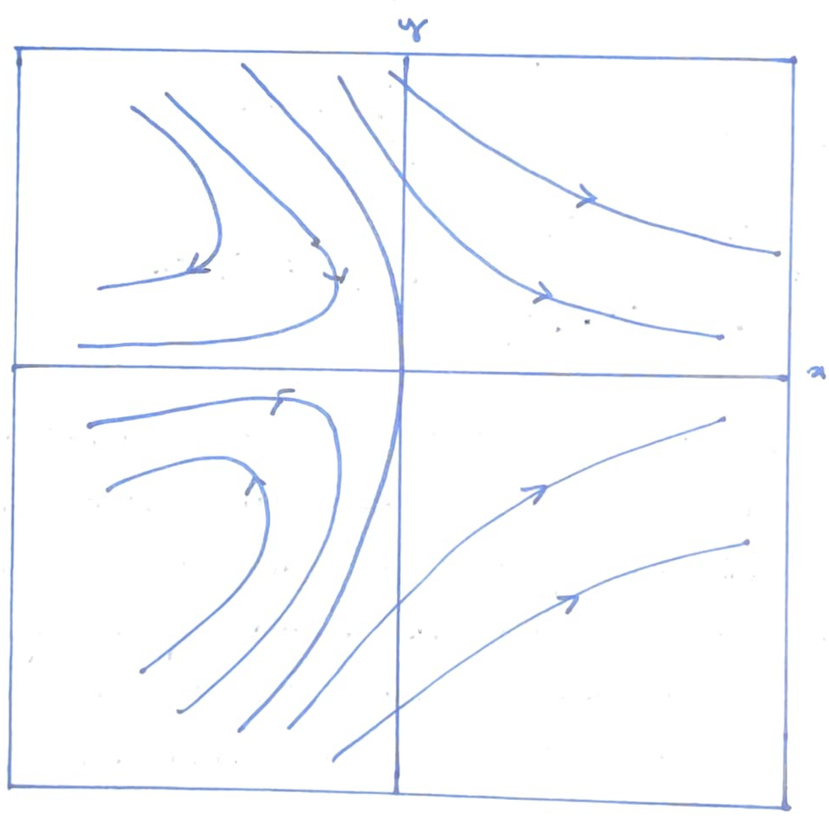
$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -y$$

the solution is $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Hence,

$x = c_1 e^{t}$ is a straight line solution. In the present case also, we see from Eq^n (6) that with $y_0 = 0$, $x = x_0 e^{t}$. The phase portrait can be drawn using the gradient

$$\frac{dy}{dx} = \frac{-y}{x + y^2}$$



From the previous example, we saw similarities between non-linear and linear analysis, especially about the equilibrium solutions. However, this may not be generalized i.e. we cannot always use linearization to determine the behaviour of a system near the equilibrium point. When can one guarantee that linearization would definitely work? In the previous example

$$\frac{dx}{dt} = x + y^2 \qquad \Leftrightarrow \qquad \frac{dx}{dt} = x$$

$$\frac{dy}{dt} = -y \qquad\qquad \frac{dy}{dt} = -y$$

— Linearized model

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \lambda = \pm 1$$

Another example would be

$$\frac{dx}{dt} = x^2 \qquad \Leftrightarrow \qquad \frac{dx}{dt} = 0$$

$$\frac{dy}{dt} = -y \qquad\qquad \frac{dy}{dt} = -y$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \lambda = 0, -1$$

In the first case, none of the eigenvalues were zero or had zero as the real part. In such a case, the equilibrium point is said to be hyperbolic and the linearized model has similar phase portrait as that of the non-linear model close to the equilibrium point. If this is not the case then linearization does not work.

We now see how to develop the linearized phase diagrams for cases where the equilibrium point is hyperbolic.

$$\frac{dx}{dt} = x^2 - y^2 - 1 \quad (= f_1, \text{ say})$$

$$\frac{dy}{dt} = 2y \quad (= f_2, \text{ say})$$

The Jacobian for the system of equations is given as

$$\underline{\underline{J}} = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$

$$\frac{\partial f_1}{\partial x} = 2x \; ; \quad \frac{\partial f_1}{\partial y} = -2y$$

$$\frac{\partial f_2}{\partial x} = 0 \; ; \quad \frac{\partial f_2}{\partial y} = 2$$

$$\Rightarrow \quad \underline{\underline{J}} = \begin{bmatrix} 2x & -2y \\ 0 & 2 \end{bmatrix}$$

We need to determine the values of x, y from equilibrium solutions to substitute in $\underline{\underline{J}}$.

$$x^2 - y^2 - 1 = 0$$
$$2y = 0$$
$$\Rightarrow \quad y = 0$$
$$\Rightarrow \quad x = \pm 1$$

Hence, the equilibrium solutions are $(1, 0)$ and $(-1, 0)$.

For $(1, 0)$, $\underline{\underline{J}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ← source

For $(-1, 0)$, $\underline{\underline{J}} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$ ← saddle

We can now analyze the bifurcations in a two state system. Before we do that, let us first consider the following system.

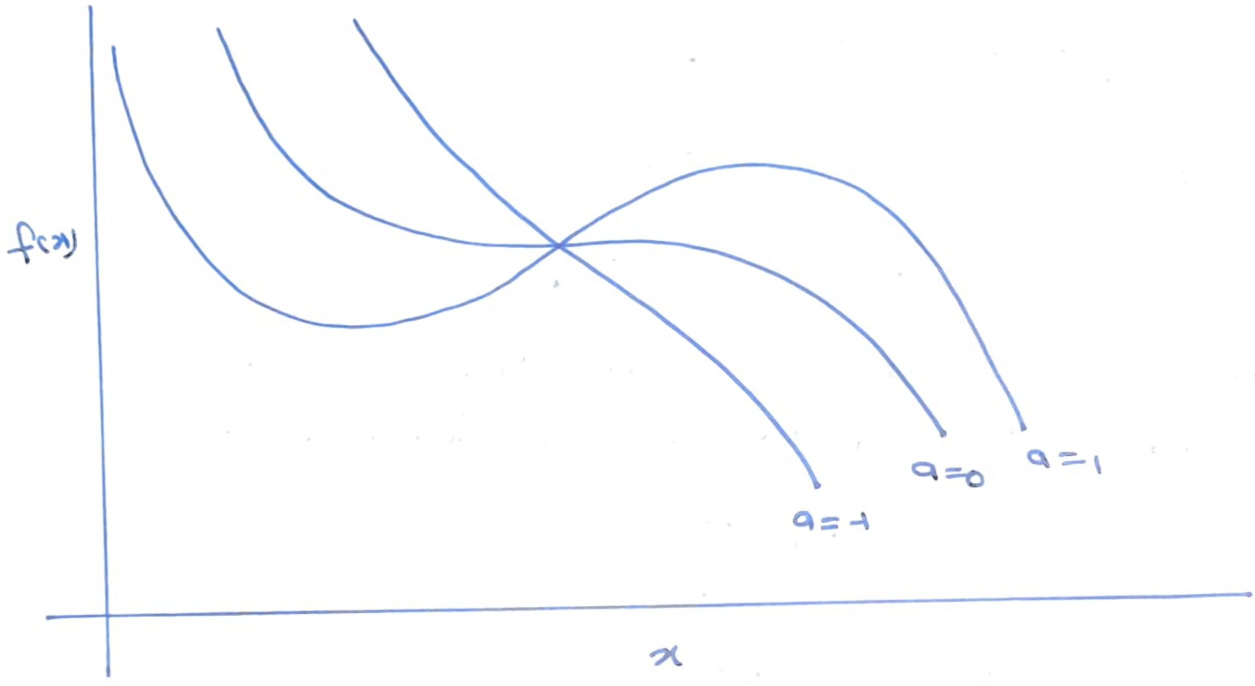
$$\frac{dx}{dt} = ax - x^3 \qquad \text{---(7)}$$

The equilibrium solutions are obtained by solving

$$f(x) = ax - x^3 = 0$$

$$\Rightarrow x = 0, \pm\sqrt{a}$$

The plot of $f(x) = 0$ as a function of a ($a = 0, \pm 1$) is given below:



For $a=-1$, $f(x)=0$ has one solution while for $a=1$, $f(x)$ has three solutions. Hence, $a=0$ is the point of bifurcation. Hence, in the above case, if $a<0$, the only equilibrium solution is $x=0$.

For a general dynamical equation

$$\frac{dx}{dt} = f(x, a)$$

the following conditions hold.

(a) Criterion for equilibrium: $\frac{dx}{dt} = f(x, a) = 0$

(b) Bifurcation point: $f(x, a) = \frac{\partial f}{\partial x} = 0$

For the present case,

$$\frac{dx}{dt} = ax - x^3 = f(x) = 0$$

$$\Rightarrow \quad x = 0, \pm\sqrt{a}$$

$$\frac{\partial f}{\partial x} = a - 3x^2 = 0$$

For a single equation, Jacobian, which is equal to the derivative, is the eigenvalue. The eigenvalue becomes zero at bifurcation point.

When $a < 0$, the only solution is $x = 0$, as seen before. Hence $\lambda = a < 0$. The system is stable.

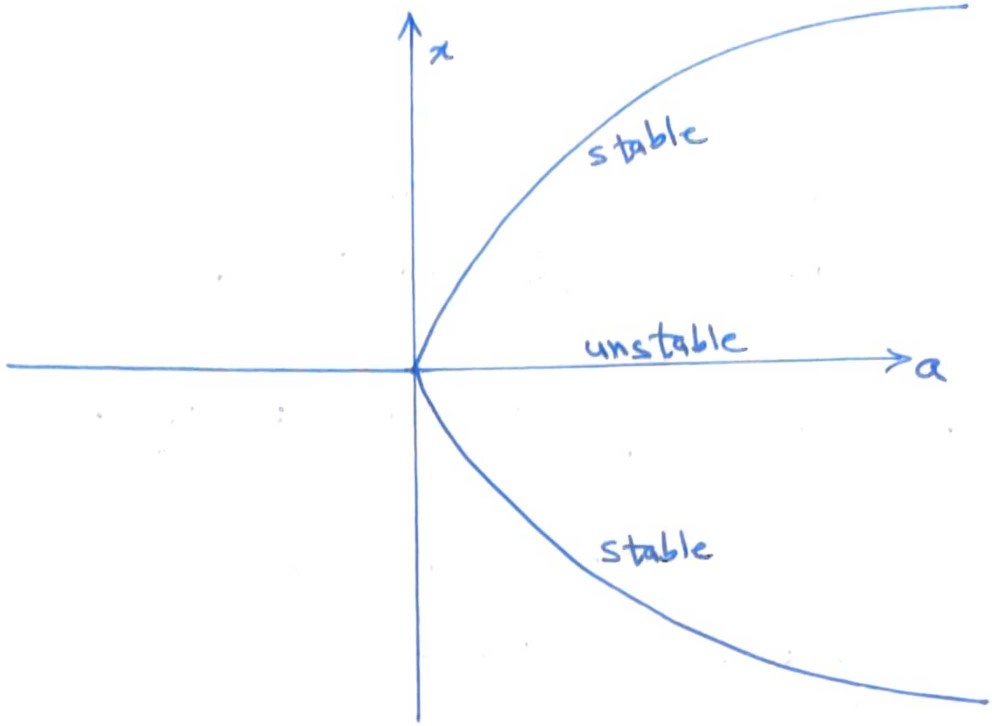
When $a > 0$, we have three solutions. These are $x = 0$, $x = \sqrt{a}$, $x = -\sqrt{a}$.

Case I: $x = 0$, $\lambda = a - 3x^2 = a > 0$ - unstable

Case II: $x = \sqrt{a}$, $\lambda = -2a$ - stable

Case III: $x = -\sqrt{a}$, $\lambda = -2a$ - stable.

Using this information, we can now develop the phase portrait.



Such type of bifurcation is called pitchfork bifurcation.

Let us now consider another system described by

$$\frac{dx}{dt} = a - x^2 \qquad - (8)$$

The equilibrium solutions are:

$$x = \pm\sqrt{a}$$

The eigenvalue can be obtained as

$$\lambda = \frac{df}{dx} = -2x$$

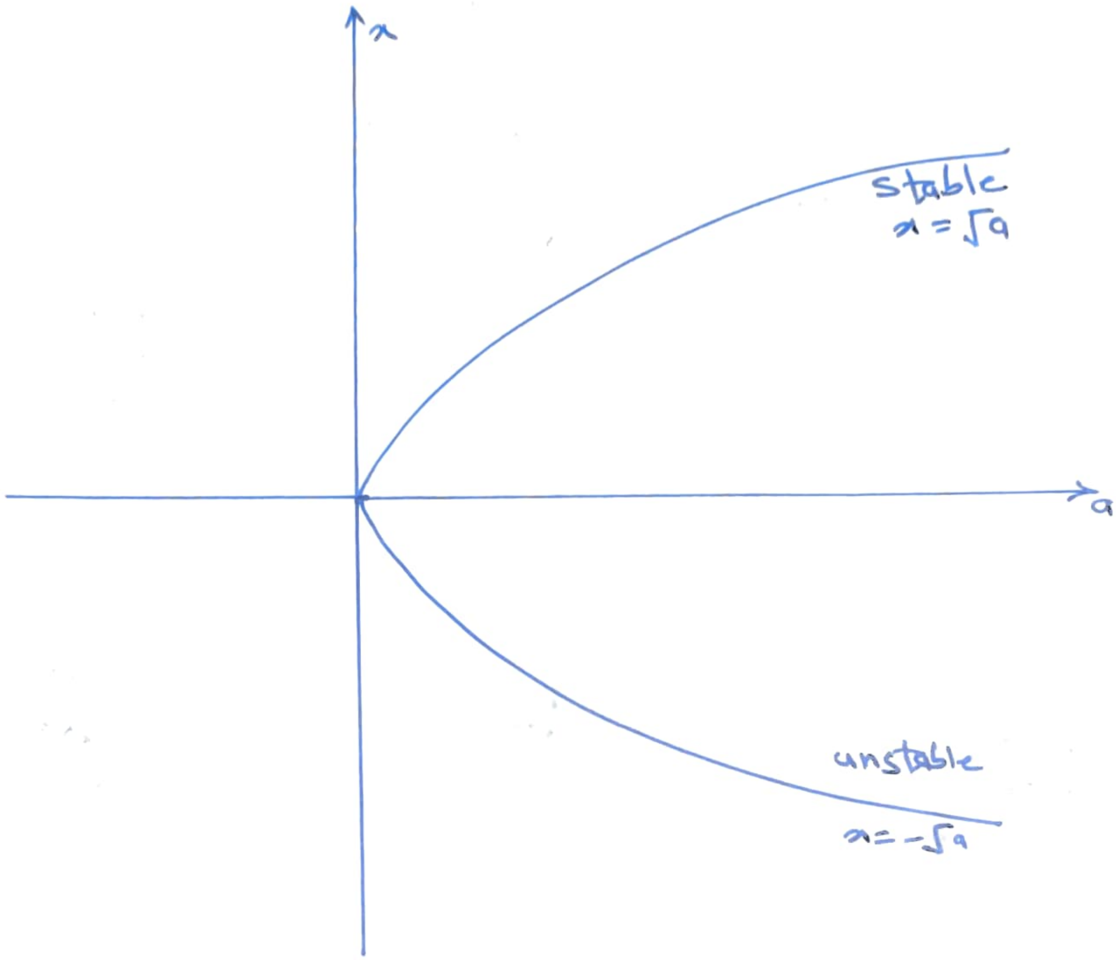
Bifurcation condition will be satisfied at $x = 0$.
Further, at bifurcation, $a - x^2 = 0 \Rightarrow a = 0$.

We now consider the case of equilibrium solutions separately.

Case I: $x = \sqrt{a}$, $\lambda = -2\sqrt{a}$ - stable

Case II: $x = -\sqrt{a}$, $\lambda = 2\sqrt{a}$ - unstable.

This puts us in a position to make the bifurcation diagram as given below.



Now we consider the third case by considering the equation

$$\frac{dx}{dt} = ax - x^2 \qquad \text{- (9)}$$

Like before, $ax - x^2 = 0$. Hence, equilibrium solutions are $x = 0, a$. The eigenvalue is given as

$$\frac{df}{dx} = \lambda = a - 2x$$

The bifurcation point can be obtained as $x = a = 0$.

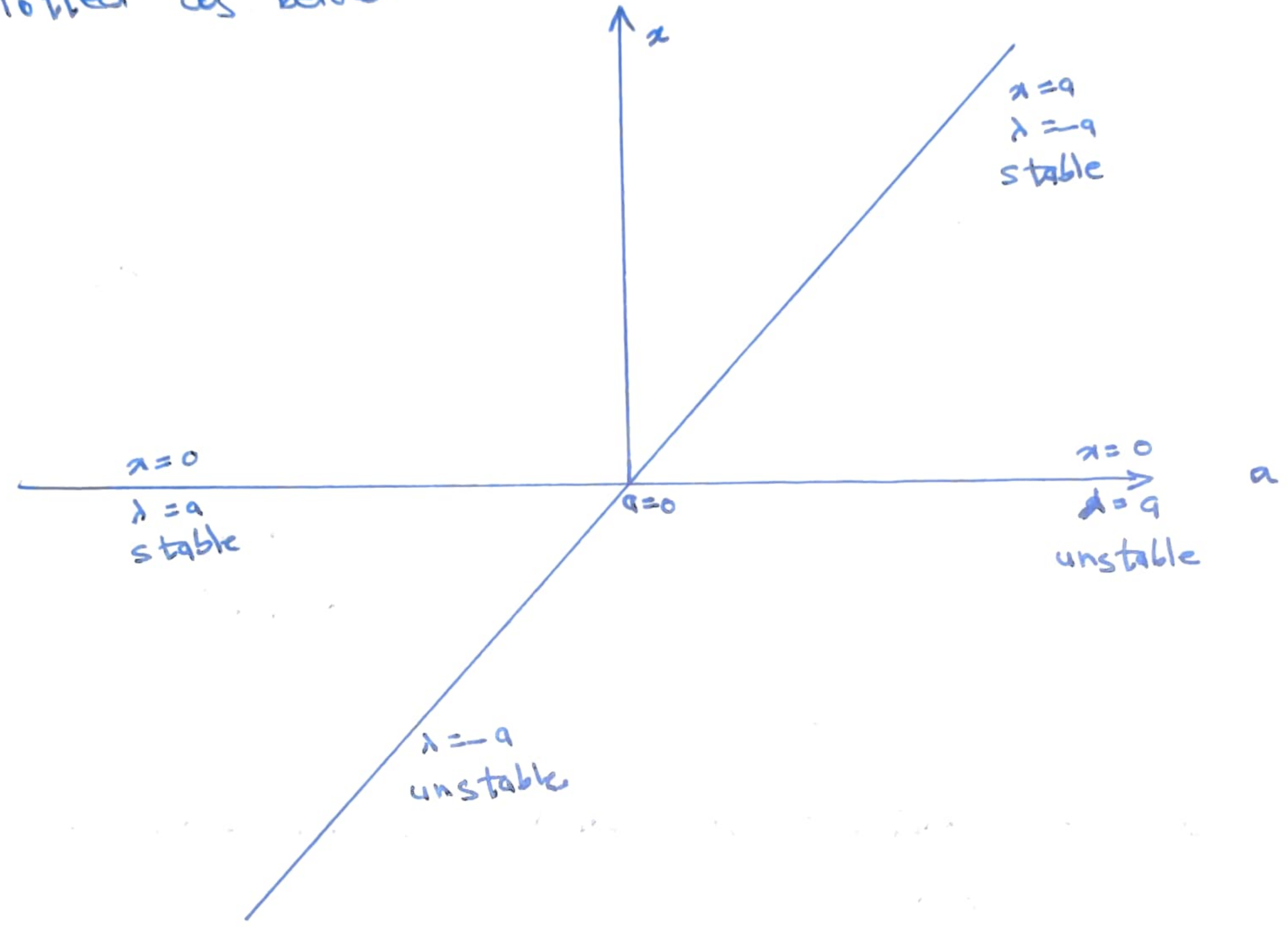
Case I: $a < 0$, $\lambda = a - 2x = a < 0$ - stable

$\lambda = a - 2x = -a > 0$ - unstable

Case II: $a > 0$, $\lambda = a - 2x = a > 0$ - unstable

$\lambda = a - 2x = -a < 0$ - stable

Using the above, the bifurcation diagram can be plotted as below.



A system exhibiting the above characteristics is said to have a transcritical bifurcation.

Now we focus our attention back to systems which are described by multiple equations.

We follow the methodology cited previously and apply the stability analysis to the following equations in polar coordinates.

$$\frac{dr}{dt} = r(1-r^2)$$

$$\frac{d\theta}{dt} = -1$$

The equations are decoupled and hence, they can be solved explicitely. The equilibrium radii are given by

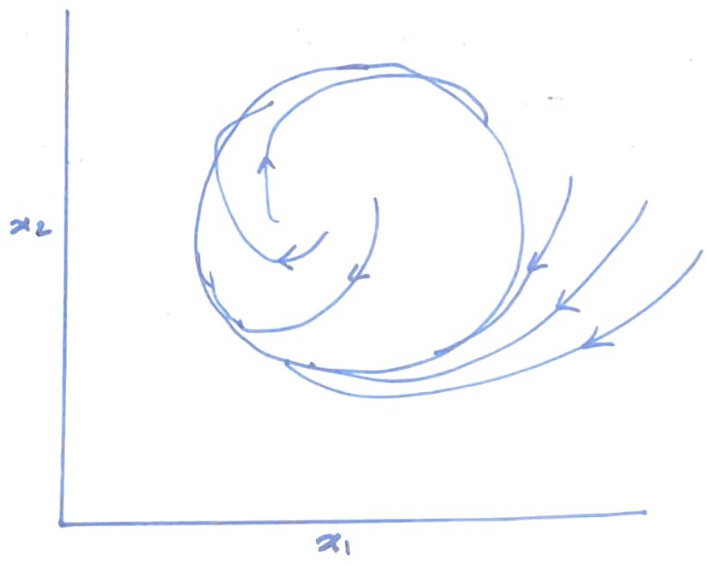
$$\frac{dr}{dt} = r(1-r^2) = 0$$

$$\Rightarrow \quad r = 0, 1.$$

The eigenvalues are obtained as

$$\frac{df}{dr} = 1 - 3r^2 = \begin{cases} 1 & \text{for } r = 0 - \text{unstable} \\ -2 & \text{for } r = 1 - \text{stable} \end{cases}$$

Hence, the "limit cycle" can be drawn as follows



A similar analysis on

$$\frac{dr}{dt} = -r(1-r^2)$$

$$\frac{d\theta}{dt} = -1$$

will show an "unstable limit cycle". Let us now consider the following equations.

$$\frac{dr}{dt} = r(a - r^2)$$

$$\frac{d\theta}{dt} = -1$$

Like before,

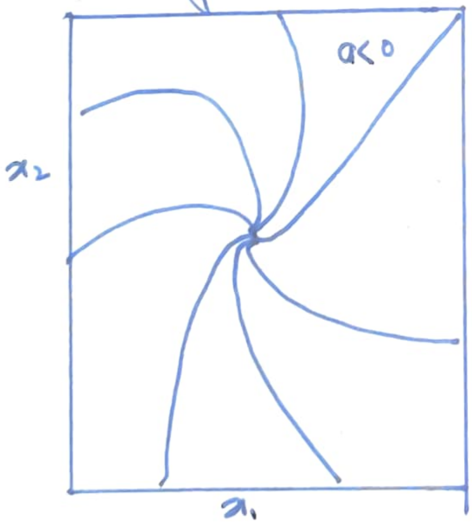
$$\frac{dr}{dt} = r(a-r^2) = 0$$

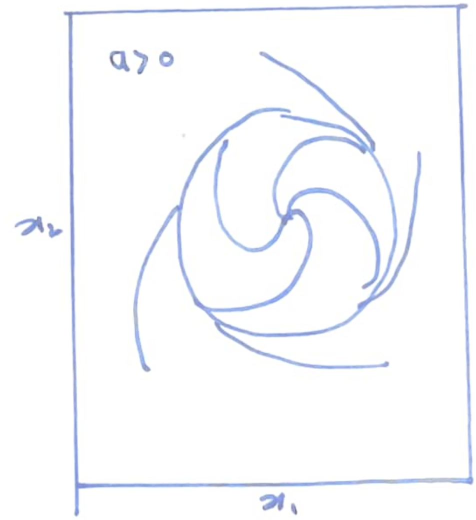
$$\Rightarrow r = 0, \sqrt{r} \qquad \text{- equilibrium solutions.}$$

$$\lambda = \frac{df}{dr} = a - 3r^2 \quad \begin{cases} a & \text{for } r = 0 \text{ - unstable} \\ -2a & \text{for } r = \sqrt{a} \text{ - stable} \end{cases}$$

for $a > 0$

$\lambda = \frac{df}{dr} = a - 3r^2 \leftarrow$ this analysis can be now done for $a < 0$, $a = 0$ and $a > 0$ to get the following bifurcation, called Hopf bifurcation.





Analysis of process dynamics in transform domain

Till now, we analyzed process systems in the state-space representation in which the evolution of the state vector was described using ordinary differential equations in time. Another way of analyzing the dynamics, especially the evolution of the system between two steady states, is by using the transform domain models. We describe this approach in detail below.

Consider a general first order system described by

$$a_1 \frac{dy}{dt} + a_0 y = b u(t) \qquad -(1)$$

$u(t)$ in the above equation is called the input variable while $y(t)$ is called the output variable. We rearrange Eq^n (1) as follows:

$$\left(\frac{a_1}{a_0}\right)\frac{dy}{dt} + y = \left(\frac{b}{a_0}\right)u(t)$$

$$\Rightarrow \quad \tau \frac{dy}{dt} + y = K u(t) \qquad -(2)$$

τ and K are called the time constant and the static or steady state gain, respectively.

At steady state,

$$\tau \frac{dy_s}{dt} + y_s = K u_s(t) \qquad -(3)$$

From Eq^{n}s (2) and (3),

$$\tau \frac{d}{dt}(y - y_s) + (y - y_s) = K(u - u_s)$$

$$\Rightarrow \tau \frac{dy^*}{dt} + y^* = K u^* \qquad -(4)$$

where y^* and u^* represent the deviation variables. Taking Laplace transform on both sides of Eq^n (4) we get

$$\tau s \bar{y}^*(s) - y^*(0) + \bar{y}^*(s) = K \bar{u}^*(s)$$

For the initial condition $y^*(0) = 0$ and suppressing $*$ throughout, we get

$$(\tau s + 1)\bar{y}(s) = K \bar{u}(s)$$

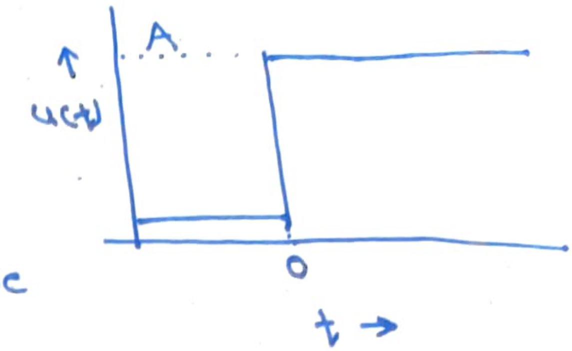
$$\Rightarrow g(s) = \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{\tau s + 1} \qquad -(5)$$

where it is understood that the Laplace transform in Eq^n (5) has been done on deviation variables. $g(s)$ is the general transfer function of a first order system.

The transfer function helps us trace the "response" of a system when it adopts to a new steady state following one or the other input which changes the steady state. We now identify different inputs and the corresponding response of a first order system.

An ideal step function:

$$u(t) = \begin{cases} 0 & t<0 \\ A & t>0 \end{cases}$$



The Laplace transform of the above function is given as

$$\bar{u}(s) = \frac{A}{s} \qquad -(6)$$

From Eq^{n}s (5) and (6)

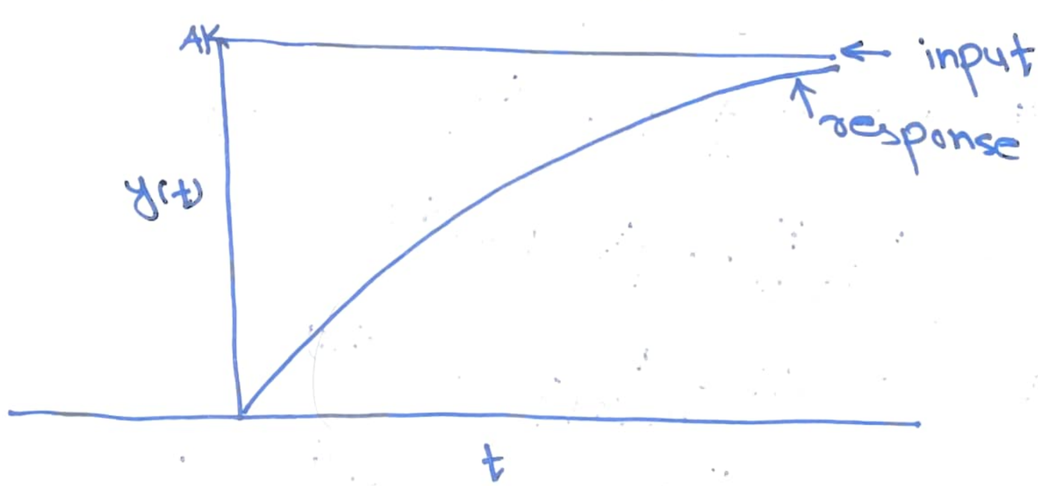
$$g(s) = \frac{K}{\tau s+1} = \frac{\bar{y}(s)}{\bar{u}(s)}$$

$$\Rightarrow \quad \bar{y}(s) = AK\left(\frac{1}{s}\right)\left(\frac{1}{\tau s+1}\right)$$

$$\Rightarrow \quad \bar{y}(s) = AK\left(\frac{1}{s} - \frac{\tau}{\tau s+1}\right)$$

$$\Rightarrow \quad y(t) = AK\left(1 - e^{-t/\tau}\right) \qquad -(7)$$

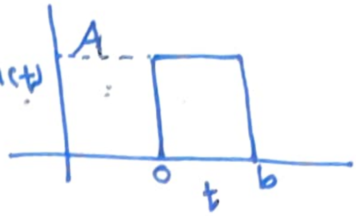
The nature of Eq^n (7) is as shown below:



The amplitude of the response will reach asymtotically a value of AK. This will happen in 4-5 time constants.

An ideal rectangular pulse response:

$$u(t) = \begin{cases} 0 & t<0 \\ A & 0<t<b \\ 0 & t>b \end{cases}$$



The Laplace transform of the above function can be determined using t-shifting rule. The function can be considered to be a combination of two step functions of magnitudes A and $-A$ acting at $t=0$ and $t=b$.

$$u(t) = A\left[H(t) - H(t-b)\right]$$

where H is the Heaviside function

$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t>0 \end{cases}$$

$$\bar{H}(s) = \frac{1}{s}$$

$$\Rightarrow \bar{u}(s) = A\left[L\left\{H(t) - H(t-b)\right\}\right]$$

$$= \frac{A}{s} - \frac{A}{s}e^{-bs}$$

$$\Rightarrow \bar{u}(s) = \frac{A}{s}(1-e^{-bs}) \qquad -(8)$$

$$\Rightarrow \bar{y}(s) = \frac{K}{\tau s+1}\cdot\frac{A}{s}(1-e^{-bs})$$

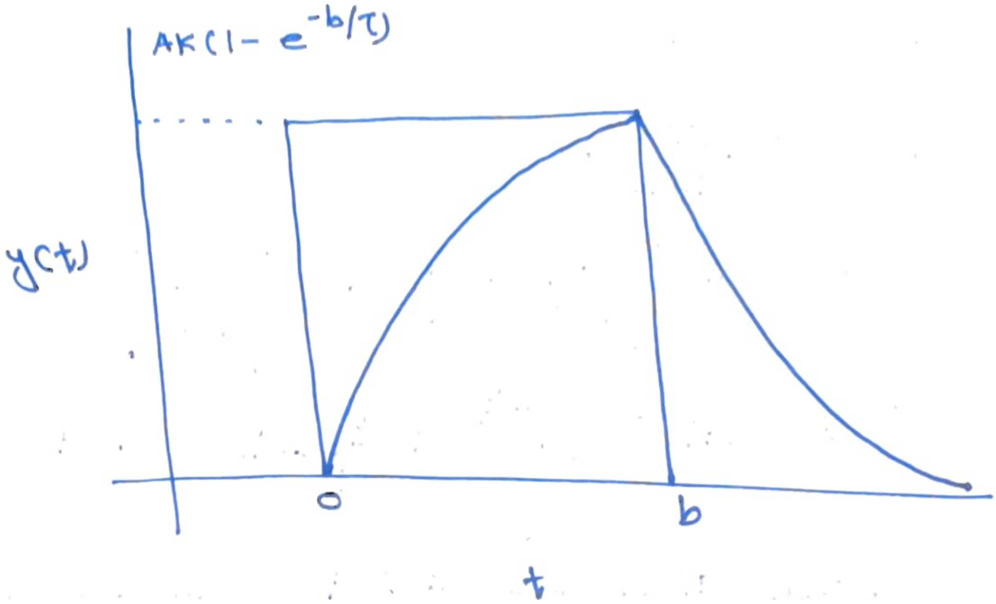
$$\Rightarrow \bar{y}(s) = AK\frac{1}{s(\tau s+1)}(1-e^{-bs})$$

$$\Rightarrow \bar{y}(s) = AK\left[\frac{1}{s(\tau s+1)} - \frac{e^{-bs}}{s(\tau s+1)}\right]$$

Inverse Laplace transformation gives

$$y(t) = \begin{cases} AK(1 - e^{-t/\tau}) & t < b \\ AK\left[(1-e^{-t/\tau}) - (1 - e^{-(t-b)/\tau})\right] & t > b \end{cases} \quad -(9)$$

The response can be plotted as shown below:



An ideal impulse response:

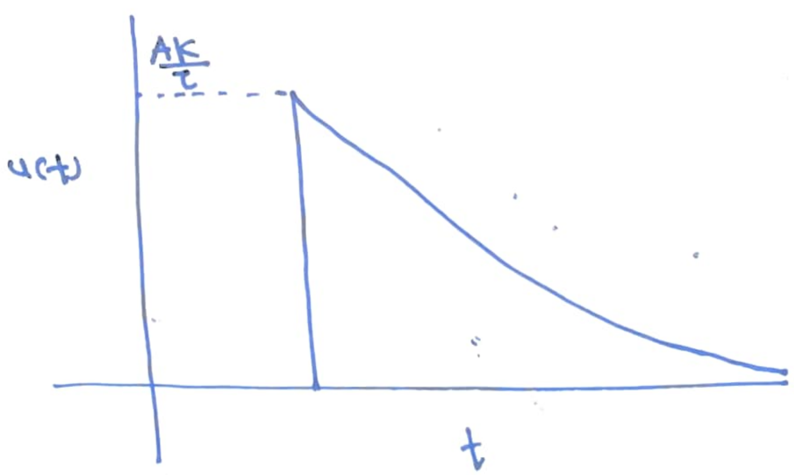
$$u(t) = A\,\delta(t) \quad -(10)$$

$$\Rightarrow \bar{u}(s) = A\,L\,\delta(t)$$

$$\Rightarrow \bar{u}(s) = A$$

$$\Rightarrow \bar{y}(s) = \frac{AK}{\tau s + 1}$$

$$\Rightarrow y(t) = \frac{AK}{\tau} e^{-t/\tau} \quad -(11)$$



An ideal ramp response:

$$u(t) = \begin{cases} 0 & t < 0 \\ At & t > 0 \end{cases}$$

$$\Rightarrow \bar{u}(s) = \frac{A}{s^2}$$

$$\Rightarrow \bar{y}(s) = \frac{AK}{(\tau s+1)s^2}$$

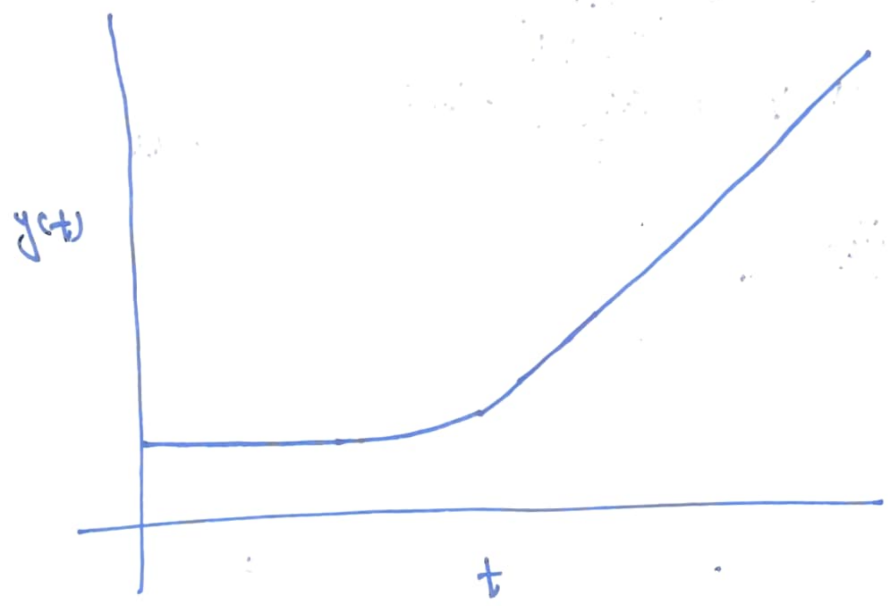
$$\Rightarrow \bar{y}(s) = AK\left[\frac{1}{s^2} - \frac{1}{s} + \frac{\tau^2}{\tau s+1}\right]$$

$$\Rightarrow y(t) = AK\tau\left(e^{-t/\tau} + \frac{t}{\tau} - 1\right) \quad - (12)$$

The response given by Eq^n (12) can be understood as follows. As $t \to \infty$, $e^{-t/\tau} \to 0$. Hence,

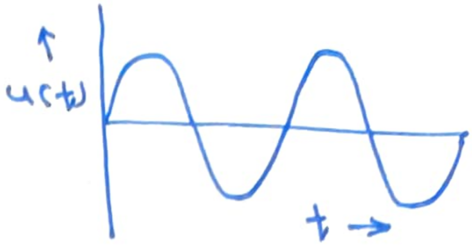
$$\lim_{t \to \infty} y(t) = AK(t - \tau)$$

Hence the slope of the response is AK and occurs at τ time displacements.



An ideal sinusoidal response:

$$u(t) = \begin{cases} 0 & t<0 \\ A\sin\omega t & t>0 \end{cases}$$



$$\Rightarrow \bar{u}(s) = \frac{A\omega}{s^2+\omega^2}$$

$$\Rightarrow \bar{y}(s) = \frac{AK\omega}{(\tau s+1)(s^2+\omega^2)}$$

$$\Rightarrow \bar{y}(s) = AK\omega\left(\frac{1}{1+\tau^2\omega^2}\right)\left(\frac{\tau}{s+1/\tau} + \frac{1}{s^2+\omega^2} - \frac{\tau s}{s^2+\omega^2}\right)$$

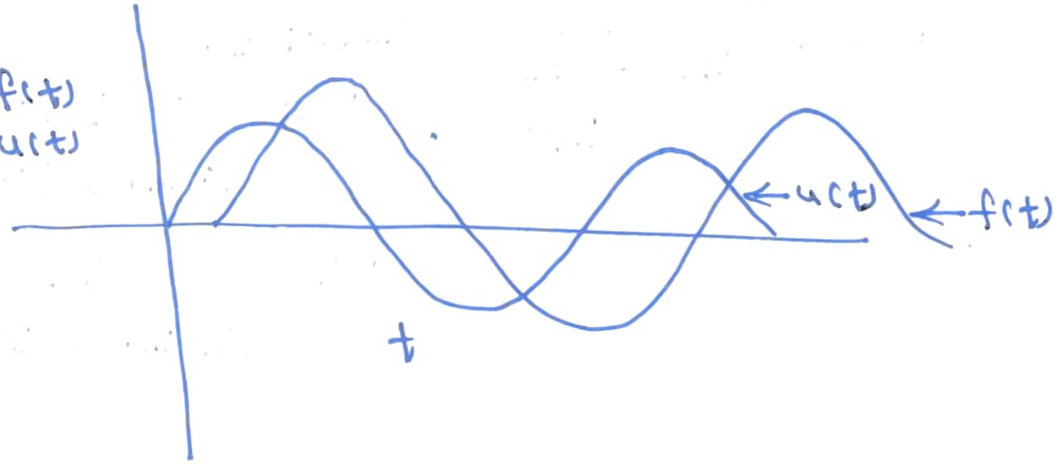
$$\Rightarrow y(t) = \left(\frac{AK\omega}{1+\tau^2\omega^2}\right)\left(\tau e^{-t/\tau} + \sin\omega t - \tau\cos\omega t\right) \quad -(13)$$

Alternatively,

$$y(t) = AK\left(\frac{\omega\tau}{1+\tau^2\omega^2} e^{-t/\tau} + \frac{1}{\sqrt{\omega^2\tau^2+1}}\sin(\omega t+\phi)\right)$$

where $\phi = \tan^{-1}(-\omega\tau)$ $\quad -(14)$

Eq^n (14) signifies that there will be transients in the response, given by $e^{-t/\tau}$ and there will be sinusoidal response at larger times, signified by $\sin(\omega t+\phi)$ with a phase lag.



The general first order transfer function, as seen before is given as

$$g(s) = \frac{K}{\tau s + 1}$$

Imagine a system which adapts quickly to the forcing function. For such a system, $\tau \sim 0$. Hence

$$g(s) = K \qquad -(15)$$

Hence, such systems have only one parameter characterizing the system which is the steady state gain K. Hence, such systems are called pure gain systems. Such systems quickly adapt to the new steady state with no transients but with gain K in the output variable. Hence, the followings can be easily derived

Step: $u(t) = \begin{cases} 0 & t<0 \\ A & t>0 \end{cases}$

$y(t) = \begin{cases} 0 & t<0 \\ AK & t>0 \end{cases}$

Pulse: $u(t) = \begin{cases} 0 & t<0 \\ A & 0<t<b \\ 0 & t>0 \end{cases}$

$y(t) = \begin{cases} 0 & t<0 \\ AK & 0<t<b \\ 0 & t>0 \end{cases}$

Impulse: $u(t) = A\delta(t)$

$y(t) = AK\delta(t)$

Ramp: $u(t) = \begin{cases} 0 & t<0 \\ At & t>0 \end{cases}$

$y(t) = \begin{cases} 0 & t<0 \\ AKt & t>0 \end{cases}$

Sinusoidal:

$u(t) = \begin{cases} 0 & t<0 \\ A\sin\omega t & t>0 \end{cases}$

$y(t) = \begin{cases} 0 & t<0 \\ AK\sin\omega t & t>0 \end{cases}$

Now we consider the case of first order dynamics

$$a_1 \frac{dy}{dt} + a_0 y = b u(t)$$

with $a_0 = 0$

$$\Rightarrow \frac{dy}{dt} = \frac{b}{a_1} u(t)$$

$$\Rightarrow \frac{dy}{dt} = K u(t) \quad ; \quad K = \frac{b}{a_1} \qquad -(16)$$

$$\Rightarrow s\bar{y}(s) = K \bar{u}(s)$$

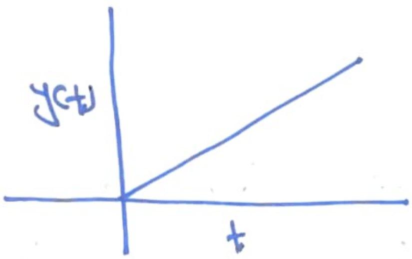
$$\Rightarrow g(s) = \frac{\bar{y}(s)}{\bar{u}(s)} = \frac{K}{s} \qquad -(17)$$

Such systems are characterized as pure capacity systems. The response of such systems to various inputs are interesting to observe.

Step response: $\bar{u}(s) = \left(\frac{1}{s}\right) A$

$$\Rightarrow \bar{y}(s) = \frac{AK}{s^2}$$

$$\Rightarrow y(t) = AKt$$

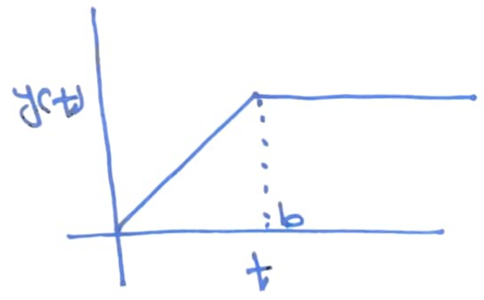


Hence, the response to a step change is a ramp response.

Rectangular pulse response:

$$\bar{y}(s) = \frac{K}{s}\left(\frac{A}{s} - \frac{A}{s} e^{-bs}\right)$$

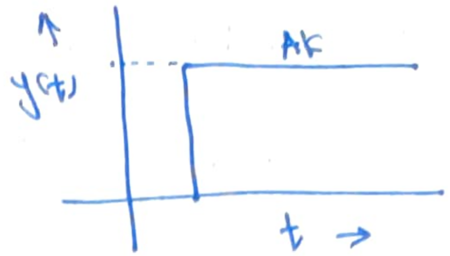
$$\Rightarrow y(t) = \begin{cases} 0 & t < 0 \\ AKt & 0 < t < b \\ AKb & t > b \end{cases}$$



Impulse response:

$$\bar{y}(s) = \frac{K}{s} \cdot A$$

$$\Rightarrow \quad y(t) = AK$$

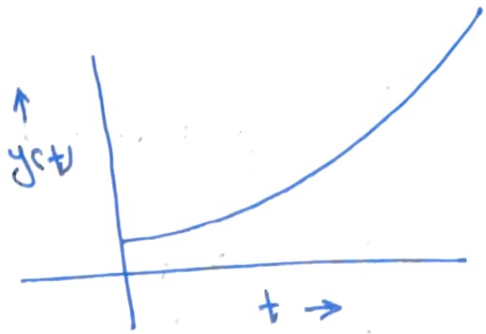


Ramp response:

$$\bar{y}(s) = \frac{K}{s} \cdot \frac{A}{s^2}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{AK}{s^3}$$

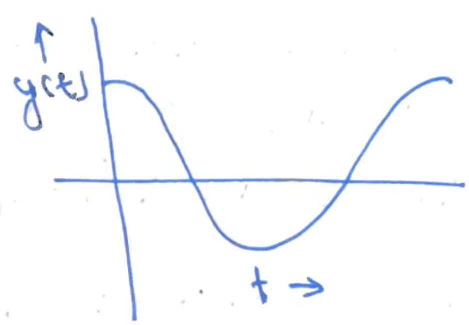
$$\Rightarrow \quad y(t) = \frac{AKt^2}{2}$$



Sinusoidal response:

$$\bar{y}(s) = \frac{AK\omega}{s(s^2+\omega^2)}$$

$$\Rightarrow \quad y(t) = \frac{AK}{\omega}(1 - \cos\omega t)$$



Hence, it can be seen that depending upon the time constant, the dynamical response of the system may drastically change.

Let us now consider a generic second order system described by the following equation:

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = b u(t) \quad -(1)$$

$$\Rightarrow \left(\frac{a_2}{a_0}\right) \frac{d^2 y}{dt} + \left(\frac{a_1}{a_0}\right) \frac{dy}{dt} + y = \left(\frac{b}{a_0}\right) u(t)$$

For $a_0 \neq 0$, we use the following standard form.

$$\frac{a_2}{a_0} = \tau^2$$

$$\frac{a_1}{a_0} = 2\xi\tau$$

$$K = \frac{b}{a_0}$$

$$\Rightarrow \quad \tau^2 \frac{d^2 y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = K u(t) \quad -(2)$$

With all initial conditions set to zero in deviation variable, the Laplace transform of the above equation gives

$$\tau^2 s^2 \bar{y}(s) + 2\xi\tau s \bar{y}(s) + \bar{y}(s) = K \bar{u}(s)$$

$$\Rightarrow g(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1} \quad -(3)$$

Eq^n (3) gives the general transfer function of the second order system described by Eq^n (1).

$\tau \rightarrow$ natural period

$\xi \rightarrow$ damping coefficient

$K \rightarrow$ steady state gain

The response of the system to a step input of magnitude A can be analyzed as follows:

$$u(t) = \begin{cases} 0 & t < 0 \\ A & t > 0 \end{cases}$$

$$\Rightarrow \bar{u}(s) = \frac{A}{s}$$

$$\Rightarrow \bar{y}(s) = \frac{AK}{s(\tau^2 s^2 + 2\xi\tau s + 1)} \quad -(4)$$

Eq^n (4) can be solved using partial fractions. Before that, we make some rearrangements.

$$\bar{y}(s) = \frac{AK/\tau^2}{s\left(s^2 + \frac{2\xi}{\tau}s + \frac{1}{\tau^2}\right)}$$

Let r_1 and r_2 be the two roots of the quadratic expression in the denominator of RHS of the above equation.

$$\bar{y}(s) = \frac{AK/\tau^2}{s(s-r_1)(s-r_2)} \quad -(5)$$

$$r_1, r_2 = -\frac{\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau} \quad -(6)$$

and $y(t) = \alpha + \beta e^{r_1 t} + \gamma e^{r_2 t}$ $\quad -(7)$

where α, β, γ are the constants to be determined with the help of partial fractions.

From Eq^n (6), it can be seen that depending upon the value of the damping coefficient ξ, r_1, r_2 may be real or imaginary, distinct or equal. The response given by Eq^n (7) will change accordingly.

Case I: $0 < \xi < 1$

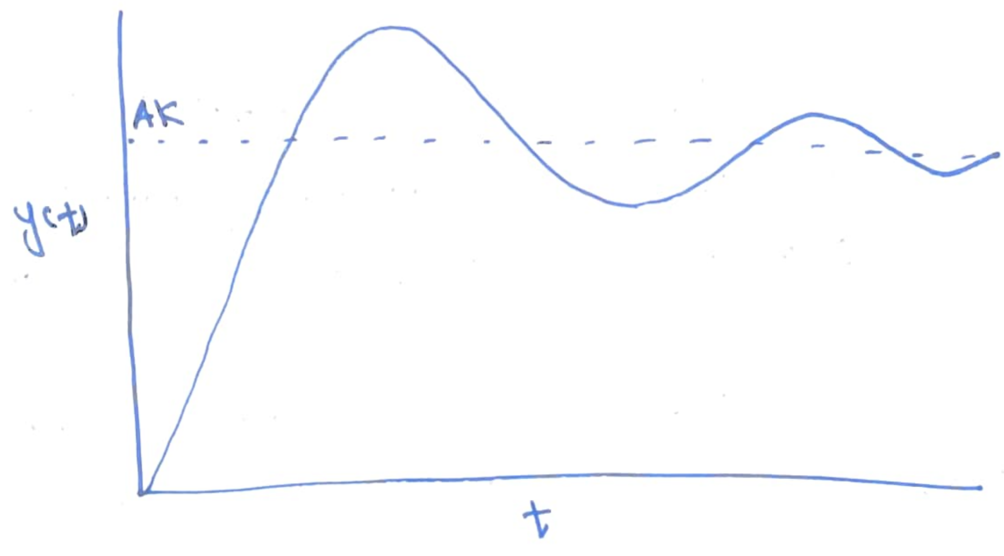
r_1, r_2 in this case will be complex numbers. A detailed solution of Eq^n (5) with partial fractions gives the following solution:

$$y(t) = AK\left[1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\frac{\xi t}{\tau}} \sin(\omega t + \phi)\right] \quad -(8)$$

where $\omega = \sqrt{1-\xi^2}/\tau$

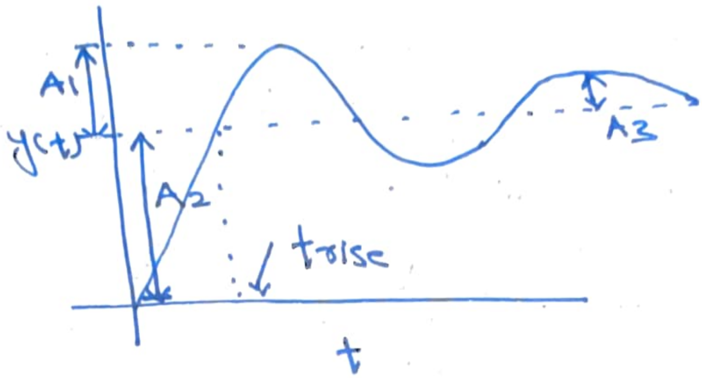
$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

The behaviour shown by Eq^n (8) is the under-damped behaviour, as signified by the exponential decay term and it is also oscillatory as signified by the sin term. The behaviour is shown below.



We analyze some features of the response shown in the previous plot.

Since the response is oscillatory, we can define rise time as the time the response becomes equal to the ultimate value for the first time.



The ratio A_1/A_2 is called the overshoot.

$$\text{overshoot} = \exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right) \qquad -(9)$$

The ratio A_3/A_1 is called the decay ratio.

$$\text{decay ratio} = \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right) \qquad -(10)$$

$$\text{decay ratio} = (\text{overshoot})^2$$

$\omega = \sqrt{1-\xi^2}/\tau$ is called the frequency of oscillation.

The period of oscillation can be obtained as

$$T = 2\pi/\omega = 2\pi\tau \Big/ \sqrt{1-\xi^2} \qquad -(11)$$

Finally, the response time is given as the time required to reach $\pm 5\%$ of the ultimate value.

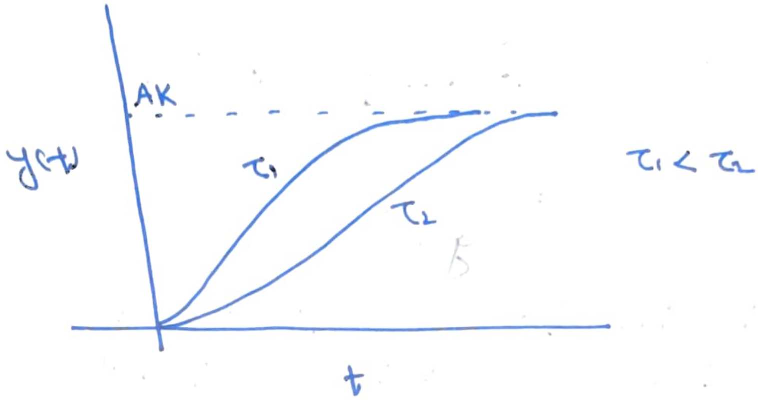
Case II: $\xi = 1$

We get $r_1 = r_2 = -1/\tau$ in this case. Hence

$$\bar{y}(s) = \frac{AK/\tau^2}{s(s-r)^2}$$

$$\Rightarrow y(t) = AK\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right] \quad -(12)$$

The behaviour of Eq^n (12) is shown below.



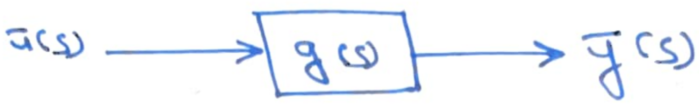
Case III: $\xi > 1$

We get real distinct roots in this case,

$$y(t) = AK\left[1 - e^{-\xi t/\tau}\left(\cosh\left(\sqrt{\xi^2-1}\right)\frac{t}{\tau} + \frac{\xi}{\sqrt{1-\xi^2}}\sinh\left(\sqrt{\xi^2-1}\ \frac{t}{\tau}\right)\right)\right] \quad -(13)$$

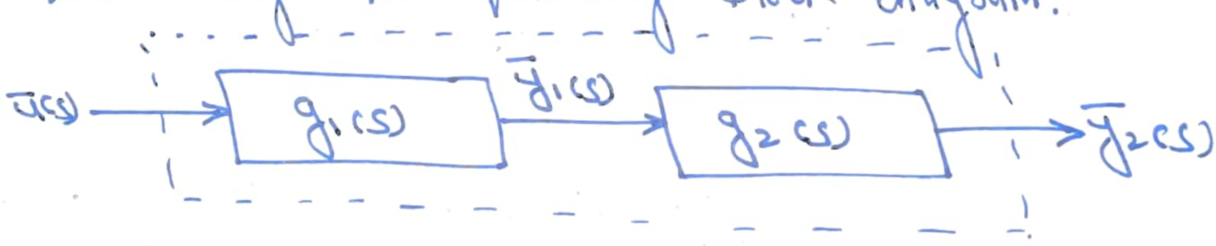
The behaviour of Eq^n (13) is similar to what is shown for $\xi = 1$ except that as ξ increases, the response keeps on becoming more and more sluggish.

Till now, we saw the response of a system which was inherently a second order system. Now we analyze dynamical systems which are second order because they comprize of two first order systems in series. In the transform domain, the input-output can be related via the transfer function, and can be represented by a block diagram as shown for a single system below.



$$g(s) = \frac{\bar{y}(s)}{\bar{u}(s)}$$

Now imagine two such first order systems in series i.e. the dynamics of individual systems is each first order. Further, the output of the first system acts as the input to the second system. The situation can be represented using the following block diagram.



The overall system dynamics has only $\bar{u}(s)$ as the input and $\bar{y}_2(s)$ as the only output. Hence, it is desired to determine the transfer function for the system. This can be done as follows.

$$g_1(s) = \frac{\bar{y}_1(s)}{\bar{u}(s)}$$

$$g_2(s) = \frac{\bar{y}_2(s)}{\bar{y}_1(s)}$$

$$\Rightarrow g_1(s)\, g_2(s) = \frac{\bar{y}_2(s)}{\bar{u}(s)} \qquad -(14)$$

Eq^n (14) gives the overall transfer function of the system. When $g_1(s)$ and $g_2(s)$ are each first order transfer functions then

$$g_1(s) = \frac{K_1}{\tau_1 s + 1}$$

$$g_2(s) = \frac{K_2}{\tau_2 s + 1}$$

$$\Rightarrow g(s) = \frac{K_1 K_2}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad -(15)$$

Hence, it can be seen that the system as a whole offers second order dynamics. When such a system is subjected to a step input then

$$\bar{y}_2(s) = \frac{A K_1 K_2}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

Using partial fractions, the above can be inverted as

$$y_2(t) = A K_1 K_2 \left[1 - \left(\frac{\tau_1}{\tau_1 - \tau_2}\right) e^{-t/\tau_1} - \left(\frac{\tau_2}{\tau_2 - \tau_1}\right) e^{-t/\tau_2}\right]$$

$$-(16)$$

When the two individual systems have equal time constants $\tau_1 = \tau_2 = \tau$ then

$$\bar{y}_2(s) = \frac{AK_1K_2}{s(\tau s+1)^2}$$

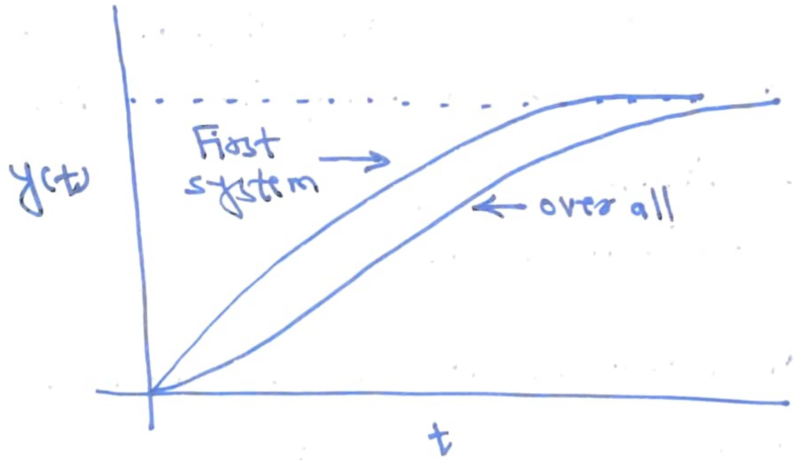
$$\Rightarrow y_2(t) = AK_1K_2\left[1 - e^{-t/\tau} - \frac{t}{\tau}e^{-t/\tau}\right] \quad -(17)$$

The response of the system for $y_2(t)$ with a unit change in $u(t)$ is given by Eq^n (17). If we have a condition that $y_1(t)$ is not affected by $y_2(t)$ then

$$g_1(s) = \frac{K_1}{\tau_1 s+1}$$

$$\Rightarrow y_1(t) = AK_1(1 - e^{-t/\tau}) \quad -(18)$$

Eq^n (18) gives the response of the first system only. Eq^{n}s (17) and (18) can be simultaneously be plotted as follows.



Hence two first order systems in series exhibit more sluggish response compared to individual first order systems.

Two first order systems were present in series in the previous case such that $y_1(t)$ was not affected by $y_2(t)$. Such systems are called non-interacting systems.

$$g(s) = \frac{K_1K_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

$$\Rightarrow g(s) = \frac{K_1K_2}{\tau_1\tau_2 s^2 + (\tau_1+\tau_2)s + 1} \quad -(19)$$

Comparing Eq^n (19) with Eq^n (3) we get

$$K = K_1K_2$$
$$\tau = \sqrt{\tau_1\tau_2}$$
$$\xi = \frac{\tau_1+\tau_2}{2\sqrt{\tau_1\tau_2}}$$

Now we consider the case of an interacting system. We consider the most general case. In interacting systems,

$$y_1 = f(y_1, y_2, u)$$
$$y_2 = g(y_1, y_2)$$

where u acts as the input to the first system while y_1 acts as the input to the second system. Conforming to the above, the most general ODE model for the system can be written as:

$$a_1 \frac{dy_1}{dt} + a_2 y_1 + a_3 y_2 = a_4 u \quad -(20)$$

$$b_1 \frac{dy_2}{dt} + b_2 y_2 - b_3 y_1 = 0 \quad -(21)$$

where Eq^{n}s (20) - (21) have been assumed to be in deviation variables.

$$\Rightarrow \frac{a_1}{a_2}\frac{dy_1}{dt} + y_1 + \frac{a_3}{a_2} y_2 = \frac{a_4}{a_2} u$$

$$\frac{b_1}{b_2}\frac{dy_2}{dt} + y_2 - \frac{b_3}{b_2} y_1 = 0$$

$$\Rightarrow \left(\frac{a_1}{a_2} s + 1\right) \bar{y}_1(s) + \frac{a_3}{a_2} \bar{y}_2(s) = \frac{a_4}{a_2} \bar{u}(s) \quad -(22)$$

$$\left(\frac{b_1}{b_2} s + 1\right) \bar{y}_2(s) - \frac{b_3}{b_2} \bar{y}_1(s) = 0 \quad -(23)$$

Eq^{n}s (22) and (23) can be solved simultaneously to give

$$\frac{\bar{y}_1(s)}{\bar{u}(s)} = \frac{\left(\dfrac{a_4 b_1 s + b_2 a_4}{a_2 b_2 + a_3 b_3}\right)}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right) s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right) s + 1} \quad -(24)$$

$$\frac{\bar{y}_2(s)}{\bar{u}(s)} = \frac{\left(\dfrac{a_4 b_3}{a_2 b_2 + a_3 b_3}\right)}{\left(\dfrac{a_1 b_1}{a_2 b_2 + a_3 b_3}\right) s^2 + \left(\dfrac{a_1 b_2 + a_2 b_1}{a_2 b_2 + a_3 b_3}\right) s + 1} \quad -(25)$$

Hence it can be seen that both the sub-systems, which otherwise followed first order dynamics, follow second order dynamics in interacting case.

The transfer function given by Eq^n (24) can be seen to be a function of 's' in the numerator also. Hence, the system is said to have a "zero" apart from the "poles" which arise out of the roots of s in the denominator. We now observe the dynamics of the system and the effect of adding zeros. Consider the following transfer function.

$$g(s) = \frac{K(x_1 s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \quad -(26)$$

From the denominator, it can be seen that the system consists of two first order systems in series. In addition, the system has a "first order lead term" in the numerator with the lead time constant as x_1. Such a system is said to be a (2, 1) order system. The response of such a system to a step input can be derived as follows:

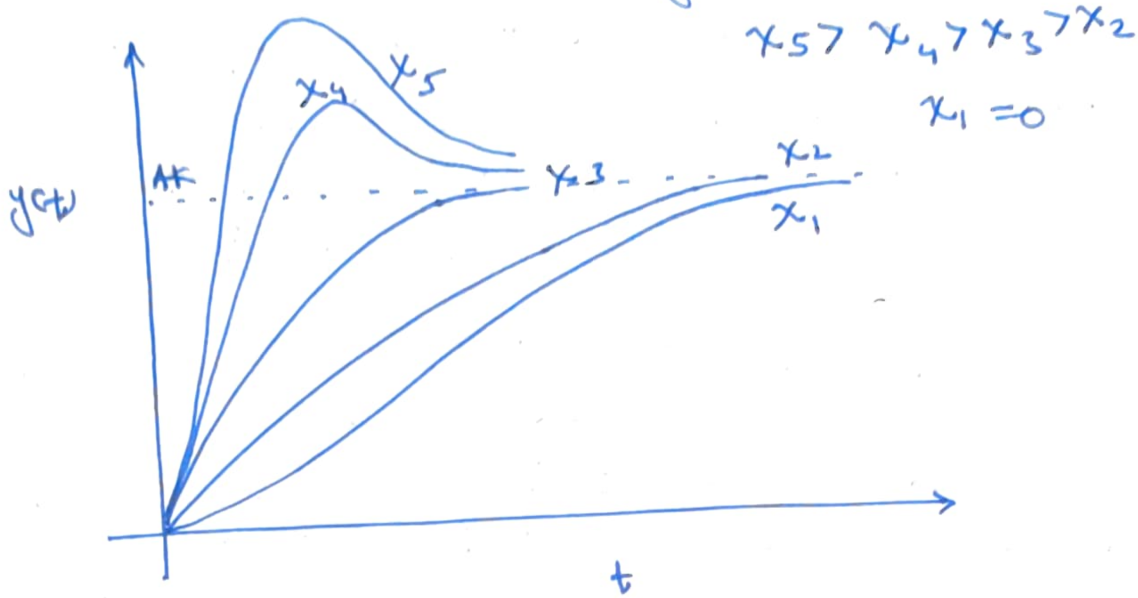
$$\bar{y}(s) = \frac{AK(x_1 s + 1)}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

Using partial fractions we obtain

$$\bar{y}(s) = AK\left[\frac{1}{s} - \frac{\tau_1(\tau_1 - x_1)}{\tau_1 - \tau_2}\frac{1}{\tau_1 s + 1} - \frac{\tau_2(\tau_2 - x_1)}{\tau_2 - \tau_1}\frac{1}{\tau_2 s + 1}\right]$$

$$\Rightarrow y(t) = AK\left[1 - \left(\frac{\tau_1 - x_1}{\tau_1 - \tau_2}\right)e^{-t/\tau_1} - \left(\frac{\tau_2 - x_1}{\tau_2 - \tau_1}\right)e^{-t/\tau_2}\right] \quad -(27)$$

The response shown by Eq^n (27) depends upon the value of x_1, the lead constant. The general behaviour is shown in the figure below.



Hence, an overshoot is expected in the system with a high lead time constant. When $x_1 = 0$, the response is same as that of two non-interacting first order systems in series.

* Overshoot happens when $x_1 >$ larger time constant
* When $x_1 = \tau_1$ or τ_2 the system exhibits first order dynamics where the effective time constant equals $\tau \neq x_1$.

To conclude, a system with higher number of p in a (p,q) will exhibit slower response while a system response will become quicker with an increase in number of q.

Continuing our discussion on (p, q) order systems, let us now consider a system in which p=1 and q=1. The general transfer function for such a system is given below.

$$g(s) = \frac{K(\chi s+1)}{(\tau s+1)} \quad - (28)$$

$$\Rightarrow g(s) = \alpha K + \frac{\beta K}{\tau s+1} = \frac{K(\chi s+1)}{(\tau s+1)}$$

The above equation can be solved to get the values of α and β as

$$\alpha = \frac{\chi}{\tau} \; ; \; \beta = 1 - \frac{\chi}{\tau}$$

$$\Rightarrow g(s) = \frac{\chi}{\tau} K + \left(1 - \frac{\chi}{\tau}\right) \frac{K}{\tau s+1} \quad - (29)$$

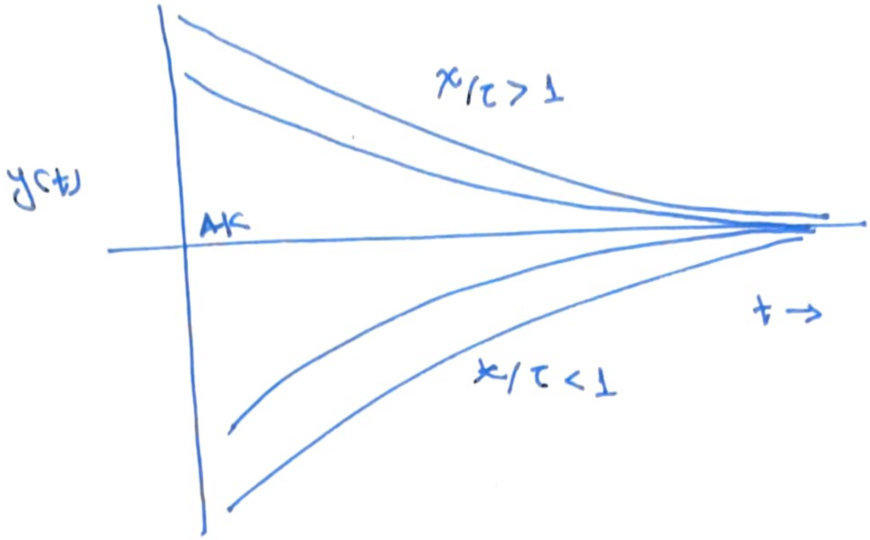
Eqn (29) can be seen to have two terms corresponding to pure gain and pure first order system. χ/τ is called lead-to-lag ratio. The response of such a system to a step change can be obtained as follows:

$$\bar{y}(s) = \frac{A\chi K}{\tau} \frac{1}{s} + \left(1 - \frac{\chi}{\tau}\right) \frac{AK}{s(\tau s+1)}$$

$$\Rightarrow y(t) = \frac{AK\chi}{\tau} + \left(1 - \frac{\chi}{\tau}\right)(1 - e^{-t/\tau})$$

$$\Rightarrow y(t) = AK\left[1 - \left(1 - \frac{\chi}{\tau}\right) e^{-t/\tau}\right] \quad - (30)$$

A typical response of such a system is shown below.



All the responses can be seen to have a non-zero value at $t = 0^+$. This is because of the pure gain term of Eq^n (29). The response reaches exponentially to AK which is governed by τ.

The initial jump or dip is characterized by x/τ ratio.

$x/\tau > 01 \rightarrow$ strong lead, initial jump

$x/\tau < 01 \rightarrow$ strong lag, initial dip

$x/\tau = 01 \rightarrow$ Pure gain behaviour

Hence, this type of (1,1) order system is said to be a lead/lag system. The previous conclusion on zeros and poles hold here. A strong zero make the response swift, hence a lead, while a strong pole makes the response sluggish, hence a lag.

The cases we considered till now were single-state, single input, single output systems. Now we consider a system with multiple inputs and multiple outputs with multiple states. The governing equations can be written as follows:

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

$$y_1 = c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1m}u_m$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2m}u_m$$

$$\vdots$$

$$y_p = c_{p1}x_1 + c_{p2}x_2 + \cdots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pm}u_m$$

The above equations, in state space representation, model a system with 'm' inputs and 'p' outputs. The system will be referred to as an 'n-state' system. The above equations can be written as matrix equations as:

$$\frac{d}{dt}\underline{X} = \underline{\underline{A}}\,\underline{X} + \underline{\underline{B}}\,\underline{U} \qquad -(1)$$

$$\underline{Y} = \underline{\underline{C}}\,\underline{X} + \underline{\underline{D}}\,\underline{U} \qquad -(2)$$

We can sequentially take the Laplace transform of the above equations to get the followings on rearrangements.

$$s\bar{x}_1 = a_{11}\bar{x}_1 + a_{12}\bar{x}_2 + \cdots + a_{1n}\bar{x}_n + b_{11}\bar{u}_1 + b_{12}\bar{u}_2 + \cdots + b_{1m}\bar{u}_m$$
$$s\bar{x}_2 = a_{21}\bar{x}_1 + a_{22}\bar{x}_2 + \cdots + a_{2n}\bar{x}_n + b_{21}\bar{u}_1 + b_{22}\bar{u}_2 + \cdots + b_{2m}\bar{u}_m$$
$$\vdots$$
$$s\bar{x}_n = a_{n1}\bar{x}_1 + a_{n2}\bar{x}_2 + \cdots + a_{nn}\bar{x}_n + b_{n1}\bar{u}_1 + b_{n2}\bar{u}_2 + \cdots + b_{nm}\bar{u}_m$$
$$\bar{y}_1 = c_{11}\bar{x}_1 + c_{12}\bar{x}_2 + \cdots + c_{1n}\bar{x}_n + d_{11}\bar{u}_1 + d_{12}\bar{u}_2 + \cdots + d_{1m}\bar{u}_m$$
$$\bar{y}_2 = c_{21}\bar{x}_1 + c_{22}\bar{x}_2 + \cdots + c_{2n}\bar{x}_n + d_{21}\bar{u}_1 + d_{22}\bar{u}_2 + \cdots + d_{2m}\bar{u}_m$$
$$\vdots$$
$$\bar{y}_p = c_{p1}\bar{x}_1 + c_{p2}\bar{x}_2 + \cdots + c_{pn}\bar{x}_n + d_{p1}\bar{u}_1 + d_{p2}\bar{u}_2 + \cdots + d_{pm}\bar{u}_m$$

$$\Rightarrow (s - a_{11})\bar{x}_1 - a_{12}\bar{x}_2 + \cdots - a_{1n}\bar{x}_n = b_{11}\bar{u}_1 + b_{12}\bar{u}_2 + \cdots + b_{1m}\bar{u}_m$$
$$-a_{21}\bar{x}_1 + (s - a_{22})\bar{x}_2 + \cdots - a_{2n}\bar{x}_n = b_{21}\bar{u}_1 + b_{22}\bar{u}_2 + \cdots + b_{2m}\bar{u}_m$$
$$\vdots$$
$$-a_{n1}\bar{x}_1 - a_{n2}\bar{x}_2 + \cdots + (s - a_{nn})\bar{x}_n = b_{n1}\bar{u}_1 + b_{n2}\bar{u}_2 + \cdots + b_{nm}\bar{u}_m$$

$$\Rightarrow \left\{ s \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \right\} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{bmatrix}$$

$$\Rightarrow (s\underline{\underline{I}} - \underline{\underline{A}})\underline{\bar{X}} = \underline{\underline{B}}\,\underline{\bar{U}}$$

$$\Rightarrow \underline{\bar{X}} = (s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}\,\underline{\bar{U}} \qquad -(3)$$

Eq^n (3) can also now be seen to be obtained directly by taking transform of Eq^n (1) treating matrices and vectors as variables.

Now we take the Laplace transform of the output variables.

$$\bar{y}_1 = c_{11}\bar{x}_1 + c_{12}\bar{x}_2 + \cdots + c_{1n}\bar{x}_n + d_{11}\bar{u}_1 + d_{12}\bar{u}_2 + \cdots + d_{1m}\bar{u}_m$$
$$\bar{y}_2 = c_{21}\bar{x}_1 + c_{22}\bar{x}_2 + \cdots + c_{2n}\bar{x}_n + d_{21}\bar{u}_1 + d_{22}\bar{u}_2 + \cdots + d_{2m}\bar{u}_m$$
$$\vdots$$
$$\bar{y}_p = c_{p1}\bar{x}_1 + c_{p2}\bar{x}_2 + \cdots + c_{pn}\bar{x}_n + d_{p1}\bar{u}_1 + d_{p2}\bar{u}_2 + \cdots + d_{pm}\bar{u}_m$$

$$\Rightarrow \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{bmatrix}$$

$$\Rightarrow \bar{Y}(s) = \underline{\underline{C}}\,\bar{\underline{X}} + \underline{\underline{D}}\,\bar{\underline{U}} \qquad -(4)$$

as can also be seen from the Laplace transform of Eq^n (2).

From Eq^n (3) and (4)

$$\bar{Y}(s) = [\underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}}]\bar{\underline{U}}(s) + \underline{\underline{D}}\,\bar{\underline{U}}(s)$$

$$\Rightarrow \bar{Y}(s) = [\underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}}]\,\bar{U}(s)$$

Hence, the matrix transfer function for the multiple input multiple output system can be written as

$$G(s) = \underline{\underline{C}}(s\underline{\underline{I}} - \underline{\underline{A}})^{-1}\underline{\underline{B}} + \underline{\underline{D}} \qquad -(5)$$

$G(s)$ is a $p \times m$ matrix where p is the number of outputs and m is the number of inputs.

Now we consider the interconversion of state-space and transform domain models and their block diagram representations. Let us first consider a single state single input single output system. The model equations in state-space domain are

$$\frac{dx}{dt} = ax + bu$$

$$y = cx + du$$

$$\Rightarrow \quad s\bar{x}(s) = a\bar{x}(s) + b\bar{u}(s)$$

$$\Rightarrow \quad \bar{x}(s) = \frac{b}{s-a}\,\bar{u}(s)$$

$$\bar{y}(s) = c\,\bar{x}(s) + d\,\bar{u}(s)$$

$$\Rightarrow \quad \bar{y}(s) = \frac{cb}{s-a}\,\bar{u}(s) + d\,\bar{u}(s)$$

$$\Rightarrow \quad \frac{\bar{y}(s)}{\bar{u}(s)} = g(s) = \frac{ds + (bc - ad)}{s-a} \qquad -(6)$$

Eqⁿ (6) the transform domain model of the system. The block diagram of the system is a simple one, as shown in the figure below.

$$\bar{u}(s) \longrightarrow \boxed{\;g(s)\;} \longrightarrow \bar{y}(s)$$

Now we convert the transform domain model and the block diagram to the state space model.

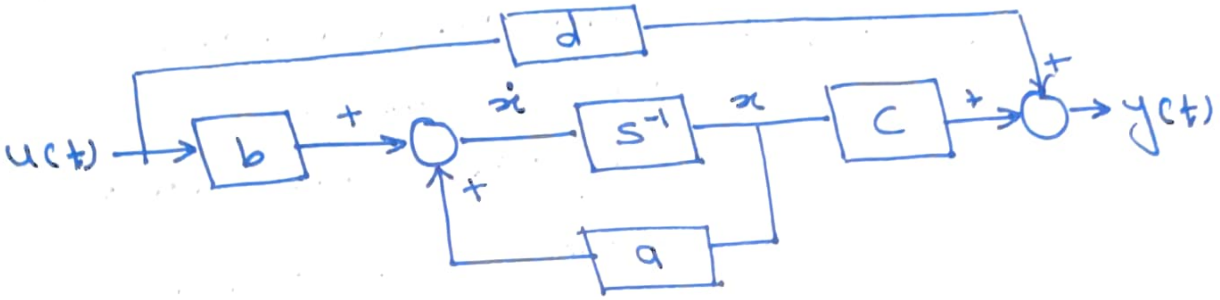
From Eq^n (6)

$$(s-a)\bar{y}(s) = [ds + (bc-ad)]\bar{u}(s)$$

$$\Rightarrow s\bar{y}(s) - a\bar{y}(s) = sd\bar{u}(s) + (bc-ad)\bar{u}(s)$$

$$\Rightarrow \frac{dy}{dt} - ay = d\frac{du}{dt} + (bc-ad)u \qquad -(7)$$

Eq^n (7) gives the state-space model. The block diagram can be drawn from the original equations as shown below.



Now we consider a second order single input single output system. The model equations are

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1u$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2u$$

$$y = c_1x_1 + c_2x_2 + du$$

The above model equations can be transformed to s-domain to get individual transfer functions $g_1(s) = \bar{x}_1(s)/\bar{u}(s)$, $g_2(s) = \bar{x}_2(s)/\bar{u}(s)$ and the overall transfer function $g(s) = \bar{y}(s)/\bar{u}(s)$.

$$(s-a_{11})\bar{x}_1 - a_{12}\bar{x}_2 = b_1\bar{u}$$

$$-a_{21}\bar{x}_1 + (s-a_{22})\bar{x}_2 = b_2\bar{u}$$

$$\Rightarrow g_2(s) = \frac{\bar{x}_2(s)}{\bar{u}(s)} = \frac{(s-a_{11})b_2 + a_{21}b_1}{(s-a_{11})(s-a_{22}) - a_{21}a_{12}} \quad -(8)$$

and $$g_1(s) = \frac{\bar{x}_1(s)}{\bar{u}(s)} = \frac{(s-a_{22})b_1 + a_{12}b_2}{(s-a_{11})(s-a_{22}) - a_{21}a_{12}} \quad -(9)$$

$$\bar{y}(s) = c_1\bar{x}_1(s) + d\,\bar{u}(s) + c_2\bar{x}_2(s)$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = g(s) = \frac{[c_1b_1(s-a_{22}) + c_1b_2a_{22} + c_2b_2(s-a_{11}) + c_2b_1a_{21} + d(s-a_{11})(s-a_{22}) - da_{12}a_{21}]}{(s-a_{11})(s-a_{22}) - a_{12}a_{21}}$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = g(s) = \frac{ds^2 + \alpha s + \beta}{s^2 + \gamma s + \delta} \quad -(10)$$

where,

$$\alpha = c_1b_1 + c_2b_2 - da_{11} - da_{22}$$

$$\beta = c_1b_2a_{22} - c_1b_1a_{22} + c_2b_1a_{21} - c_2b_2a_{11} + da_{11}a_{22} - da_{21}a_{12}$$

$$\gamma = -a_{11} - a_{22}$$

$$\delta = -a_{21}a_{22}$$

The state space domain model for the system with single input and single output can be derived as follows:

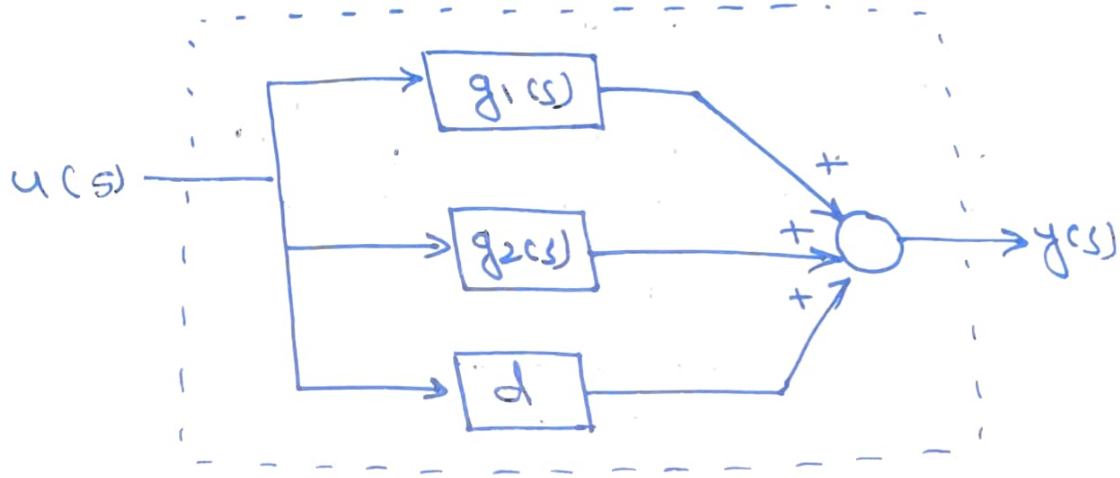
From the transfer function of Eq^n (10),

$$s^2 \bar{y}(s) + s\gamma\bar{y}(s) + \delta\bar{y}(s)$$
$$= s^2 d\,\bar{u}(s) + s\alpha\bar{u}(s) + \beta\bar{u}(s)$$

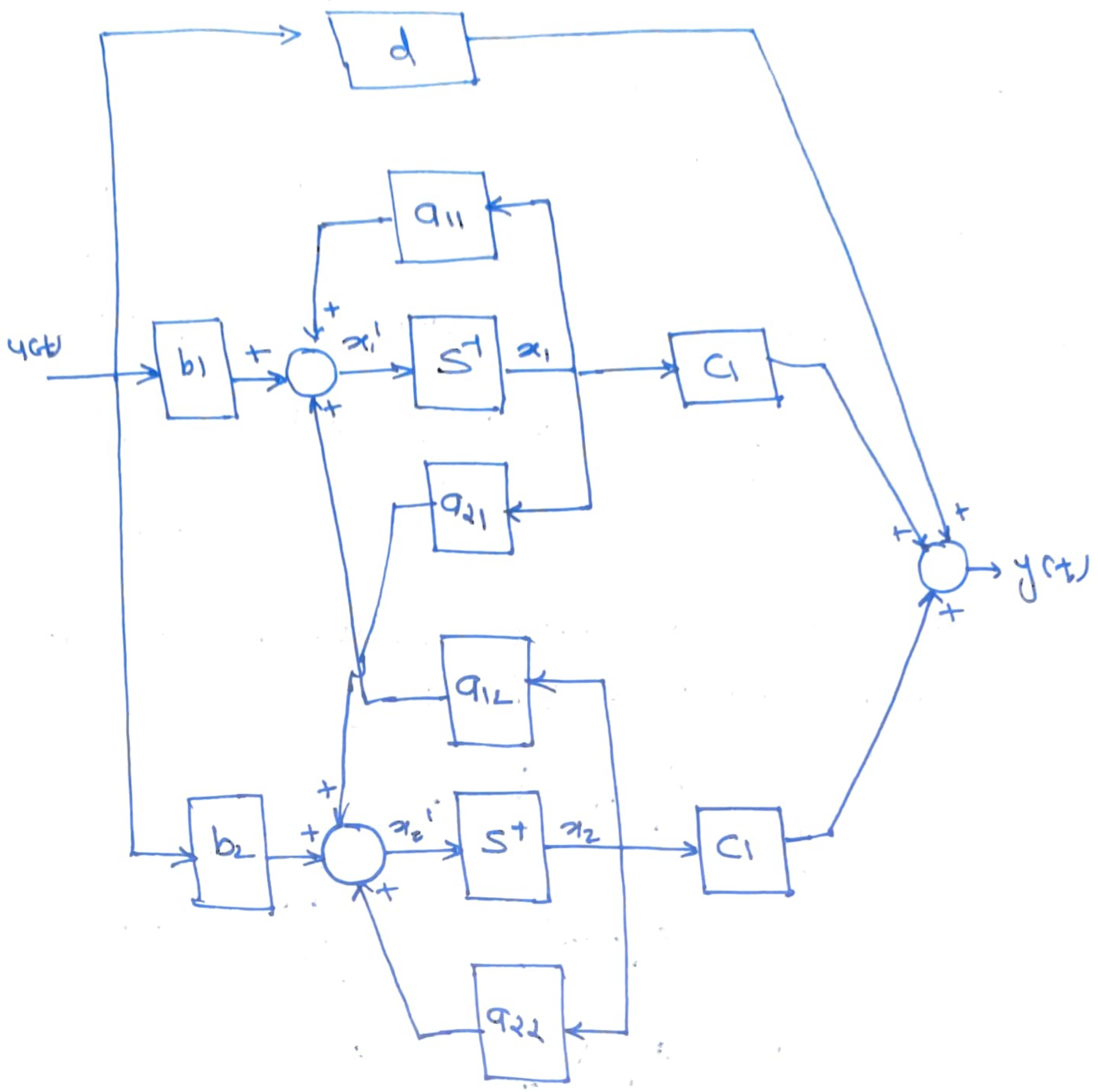
Assuming all initial conditions to be zero, the above equation can be inverted as follows:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \delta y = \frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + \beta u \quad -(11)$$

The transform domain block diagram for the system can be drawn as follows:



The above block diagram shows the additive effects coming from the effect of x_1 via the transfer function $g_1(s)$, effect of x_2 via the transfer function $g_2(s)$ and of the input variable $u(s)$ directly as d. The state-space domain block diagram can be drawn as shown.



Following the above principles it is possible to convert state-space models to single input single output state-space and transform domain models and the corresponding block diagrams can be drawn.

Till now we saw the effect of poles and zeros on the dynamical response of a system. We also found the overall transfer function of a systems which were not single state or single input single output. Let us now consider the following systems in series.

$$g_1(s) = \frac{-2s+1}{5s+1} \quad -(12)$$

$$g_2(s) = \frac{1}{-2s+1} \quad -(13)$$

The overall transfer function is given as

$$g(s) = g_1(s)\, g_2(s) = \frac{1}{5s+1} \quad -(14)$$

Hence, the zero of $g_1(s)$ cancells the pole of $g_2(s)$. Since x and τ are parameters which may not be accurately known, let us imagine a small error in $g_2(s)$ as

$$g'_2(s) = \frac{1}{-2.0001s+1}$$

$$\Rightarrow g'(s) = \frac{(-2s+1)}{5s+1} \times \frac{1}{-2.0001s+1}$$

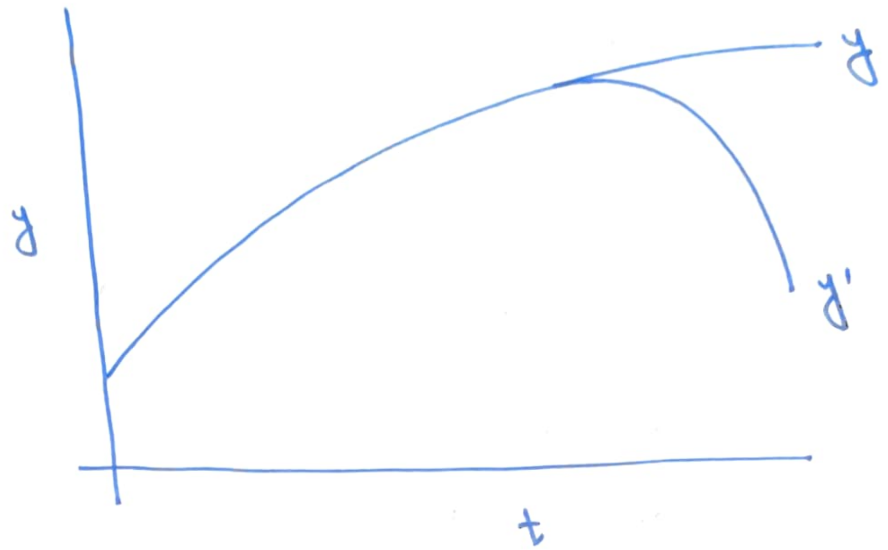
$$\Rightarrow g'(s) = \frac{-2s+1}{-10.0005s^2+2.9999s+1} \quad -(15)$$

The step responses given by Eq^n (14) and (15) can be compared.

$$y(t) = 1 - e^{-t/5} \quad -(16)$$

$$y'(t) = 1 - \frac{7}{7.0001} e^{-t/5} - \frac{0.0001}{7.0001} e^{t/2.0001} \quad -(17)$$

Eq^n (16) and (17) can be plotted as below.



y' doesnot represent the derivative. Rather, it represents the response following $g'(s)$. Hence, it can be seen that the introduction of small error in systems exhibiting pole-zero cancellation can lead to a very different fate of the system. This highlights the importance of accurate determination of the parameters of the system.

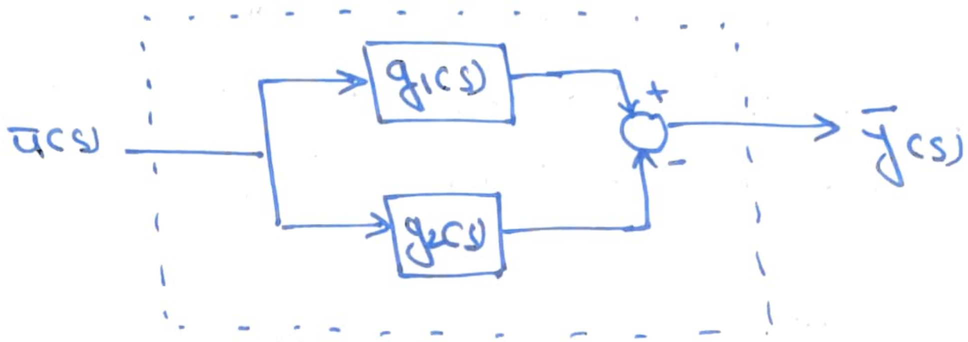
The instability of the system was not very apparent in this case when the system was analyzed in transform domain. The same can be easily seen when we cast the problem in state-space domain

Let us now consider an overall transfer function as given below.

$$g(s) = g_1(s) - g_2(s) \quad -(1)$$

$$g(s) = \frac{K_1}{\tau_1 s+1} - \frac{K_2}{\tau_2 s+1} \quad ; K_1 > K_2$$

The first mode, $g_1(s)$, is called the main mode while the second mode, $g_2(s)$, is called the opposition mode. The block diagram of the system is as shown below.



Let us consider the step response of such a system.

$$\bar{y}(s) = g_1(s)\bar{u}(s) - g_2(s)\bar{u}(s)$$

$$\Rightarrow \bar{y}(s) = \frac{AK_1}{s(\tau_1 s+1)} - \frac{AK_2}{s(\tau_2 s+1)}$$

$$\Rightarrow y(t) = AK_1(1-e^{-t/\tau_1}) - AK_2(1-e^{-t/\tau_2}) \quad -(2)$$

From Eq^n (2), $\lim_{t\to\infty} y(t) = A(K_1 - K_2)$. Also, it can be seen that

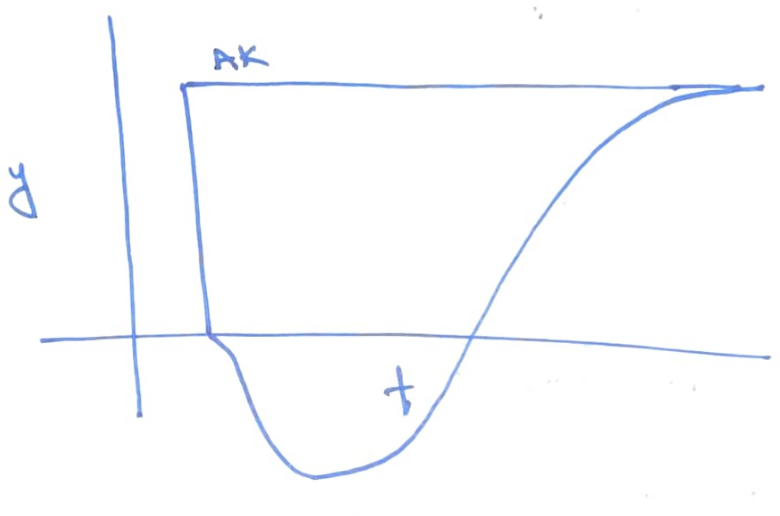
$$y = y_1 - y_2$$

where y_1 and y_2 are individual responses corresponding to the main and opposition modes.

$$\frac{dy}{dt} = \frac{dy_1}{dt} - \frac{dy_2}{dt}$$

$$\Rightarrow \left.\frac{dy}{dt}\right|_{t=0} = \frac{K_1}{\tau_1} - \frac{K_2}{\tau_2} \quad \text{--- (3)}$$

Eq^n (3) gives the response derivative at time $t=0$. It is interesting to note that the sign of this derivative can be positive or negative. For $K_2/\tau_2 > K_1/\tau_2$, $dy/dt|_{t=0} < 0$. Hence the system exhibits an "inverse response".



The above response happens because the opposition gain is smaller than main gain and the opposition mode time constant is small so as to have the system act quickly in favour of the opposition mode.

Let us now simplify the overall transfer function.

$$g(s) = \frac{K_1(\tau_2 s + 1) - K_2(\tau_1 s + 1)}{(\tau_2 s + 1)(\tau_1 s + 1)}$$

$$\Rightarrow g(s) = \frac{(K_1\tau_2 - K_2\tau_1)s + (K_1 - K_2)}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$\Rightarrow g(s) = \frac{K(x s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad - (4)$$

where $K = K_1 - K_2$

$$x = \frac{K_1\tau_2 - K_2\tau_1}{K_1 - K_2}$$

Eq^n (4) can be identified as the transfer of a (2, 1) order system. For the conditions of inverse response,

$$K_1 > K_2$$

$$\frac{K_1}{\tau_1} < \frac{K_2}{\tau_2}$$

$$K_1 > 0 \quad K_2 > 0$$

the value of x can be seen to be negative. Hence, an inverse response system is a system with a positive zero. We now analyze the response of this generalized system to a step input.

Consider a general (2, 1) order transfer function satisfying the conditions of inverse response.

$$g(s) = \frac{K(-\eta s + 1)}{(\tau_1 s + 1)(\tau_2 s + 1)} \qquad -(5)$$

The response of this system subject to a step input is given as follows:

$$\bar{y}(s) = \frac{AK(-\eta s + 1)}{s(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$\Rightarrow \bar{y}(s) = AK\left[\frac{1}{s} + \frac{\tau_1(\tau_1 + \eta)}{\tau_2 - \tau_1}\,\frac{1}{\tau_1 s + 1} + \frac{\tau_2(\tau_2 + \eta)}{\tau_1 - \tau_2}\,\frac{1}{\tau_2 s + 1}\right]$$

$$\Rightarrow y(t) = AK\left[1 + \left(\frac{\eta + \tau_1}{\tau_2 - \tau_1}\right)e^{-t/\tau_1} + \left(\frac{\eta + \tau_2}{\tau_1 - \tau_2}\right)e^{-t/\tau_2}\right] \qquad -(6)$$

$$\Rightarrow \lim_{t \to \infty} y(t) = AK$$

Hence, the ultimate response of the system equals the gain times the magnitude of the step input.

$$\frac{dy}{dt} = \frac{AK}{\tau_1 \tau_2}\left[\tau_2\left(\frac{\eta + \tau_1}{\tau_1 - \tau_2}\right)e^{-t/\tau_1} + \tau_1\left(\frac{\eta + \tau_2}{\tau_2 - \tau_1}\right)e^{-t/\tau_2}\right]$$

$$\lim_{t \to 0} \frac{dy}{dt} = \frac{-AK\eta}{\tau_1 \tau_2} \qquad -(7)$$

Since all the quantities on RHS of Eq^n (7) are positive, we get $\lim_{t \to 0} \frac{dy}{dt} < 0$ i.e. we observe inverse response. The time of inversion of response can be obtained by setting $dy/dt = 0$.

$$t_{inversion} = \frac{\tau_1 \tau_2}{\tau_1 - \tau_2} \ln\left[\frac{\tau_1(\eta + \tau_2)}{\tau_2(\eta + \tau_1)}\right] \quad -(8)$$

The second derivative can be determined as

$$\frac{d^2y}{dt^2} = \frac{AK}{(\tau_1\tau_2)^2}\left[\frac{\tau_2^2(\eta+\tau_1)}{\tau_2-\tau_1} e^{-t/\tau_1} - \frac{\tau_1^2(\eta+\tau_2)}{\tau_2-\tau_1} e^{-t/\tau_2}\right] \quad -(9)$$

The above equation can be simplified after substitution of $t_{inversion}$ from Eq^n (8) to observe that d^2y/dt^2 is positive indicating minima.

A few concluding remarks can be written from the analyses carried out till now.

* Effect of adding zeros and poles to an inverse response system remains qualitatively the same i.e. addition of zero makes the response faster while addition of a pole makes the response slower, keeping the overall features of inverse response the same.

* It is seen that addition of one positive zero makes the system exhibit inverse response. With the addition of more positive zeros, the system exhibits multiple minima/maxima. Initial inverse response is observed for systems with odd number of positive poles and for the system with even number, initial positive response is observed.

In all the transfer function cases that we analyzed till now, the system started responding to the input immediately. The pace to reach the new steady state varied depending upon the transfer function. However there can be situations where the system starts responding to an input after a finite amount of time. Such systems are referred to as time-delay systems. For such processes the transfer function must be augmented by $e^{-\alpha s}$ term where α is the dead time.

We previously saw the case of a pure gain system in which there were no transient dynamics. The output was simply multiplied by the gain K. For a unit gain, the pure gain transfer function would be

$$g(s) = 1 \qquad -(10)$$

Eq^n (10) simply means that the output will be identical to the input and the system would respond instantaneously. Now if we want the output to be identical to the input but only shifted by α time units then we get a pure time-delay transfer function which is as given below.

$$g(s) = e^{-\alpha s} \qquad -(11)$$

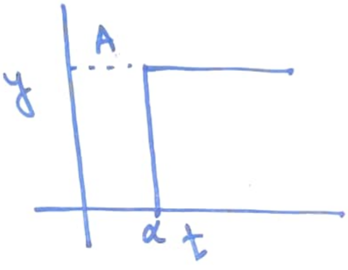
All the transfer functions considered previously were ratios of a q-th order numerator polynomial and a p-th order denominator polynomial. However, Eq^n (11) does not seem to look like that. But we can reach a similar form as shown below.

$$g(s) = e^{-\alpha s}$$

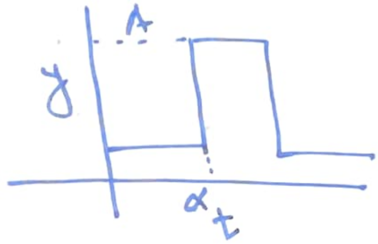
$$\Rightarrow g(s) = \frac{1}{e^{\alpha s}}$$

$$\Rightarrow g(s) = \frac{1}{1 + \alpha s + \frac{\alpha^2 s^2}{2!} + \frac{\alpha^3 s^3}{3!} + \dots} \quad -(12)$$

Hence a pure time-delay system can be identified as an infinite order system. The response of a pure time-delay system can be seen as below.

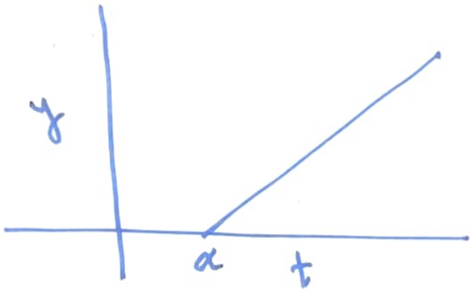


step response



pulse response

impulse response



ramp response

sinusoidal response
$\phi = \alpha\omega$

Consider a system comprising of N non-interacting first order systems in series, each having a time constant of α/N. The overall transfer function for such a system is given as

$$g(s) = \frac{1}{\left(\frac{\alpha}{N}s + 1\right)^N} \quad - (13)$$

where the overall gain has been assumed to be 1.

We have

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e$$

$$\Rightarrow \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{-\alpha m} = e^{-\alpha m}$$

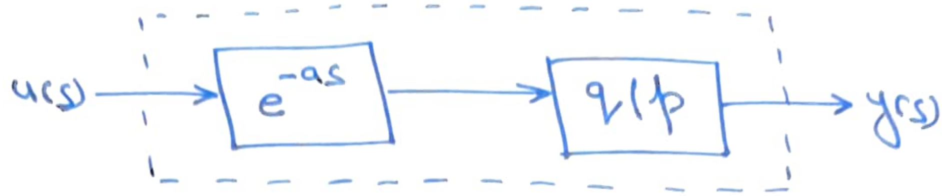
$$\Rightarrow e^{-\alpha m} = \lim_{m\to\infty}\frac{1}{\left(1+\frac{1}{m}\right)^{\alpha m}}$$

$$\Rightarrow e^{-\alpha} = \lim_{N\to\infty}\frac{1}{\left(1+\frac{\alpha}{N}\right)^N} \quad ; \text{ assuming } \alpha m = N$$

$$\Rightarrow \lim_{N\to\infty} g(s) = e^{-\alpha s} \quad - (14)$$

It can be seen from Eqⁿs (13) and (14) that the time-delay system acts as the limiting case of infinite number of non-interacting first order system, each having a time constant of α/N, connected in series.

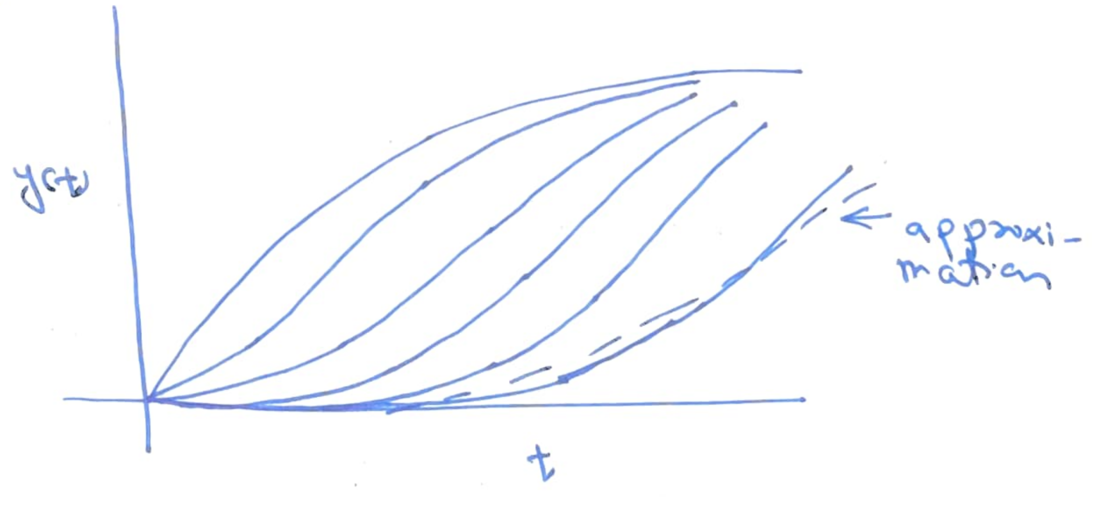
A general time-delay system can be identified as the one with a transfer function of i/p form augmented with an $e^{-\alpha s}$ term. Hence the system can be represented by a block diagram as shown below.



The response of such a system can be obtained by inverting the Laplace transform of the i/p part and shifting the response by α time units.

$$y(t) = \begin{cases} 0 & t < \alpha \\ f(t-\alpha) & t > \alpha \end{cases} \qquad -(15)$$

We saw previously that with an increase in the order of a system, the response becomes more sluggish. Hence higher order systems can be approximated as first order with time-delay as shown below.



Time-delay transfer function can also be converted to a q/p form using Padé approximation. According to this approximation,

$$e^{-\alpha s} = \frac{a_q(\alpha s)^q + a_{q-1}(\alpha s)^{q-1} + \cdots + a_0}{b_p(\alpha s)^p + b_{p-1}(\alpha s)^{p-1} + \cdots + b_0} \quad -(16)$$

The first order approximation can be written as

$$e^{-\alpha s} = \frac{1 - \frac{\alpha}{2}s}{1 + \frac{\alpha}{2}s} \quad (p=q=1) \quad -(17)$$

Similarly, the second and third order approximations are written as

$$e^{-\alpha s} = \frac{1 - \frac{\alpha}{2}s + \frac{\alpha^2}{12}s^2}{1 + \frac{\alpha}{2}s + \frac{\alpha^2}{12}s^2} \quad (p=q=2) \quad -(18)$$

$$e^{-\alpha s} = \frac{1 - \frac{\alpha}{2}s + \frac{\alpha^2}{10}s^2 - \frac{\alpha^3}{120}s^3}{1 + \frac{\alpha}{2}s + \frac{\alpha^2}{10}s^2 + \frac{\alpha^3}{120}s^3} \quad -(19)$$

To conclude, to handle a process with time-delay we need to solve for the transfer function model in a usual way without the time-delay. Then, a shift in the response by α time units has to be made. A utility of time-delay systems could be clearly seen as the approximation of higher order system response as first order response with a time-delay.

While analyzing the stability of a system in state-space domain, we made use of the eigen-values of the system to decide upon the stability. Now we analyze the stability of the systems modelled in transform domain.

If $g(s)$ is the transfer function of a system then

$$\bar{y}(s) = g(s)\,\bar{u}(s) \qquad - (20)$$

For a (p, q) order system,

$$g(s) = \frac{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_0}{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_0} \qquad - (21)$$

If each of the polynomials in Eq^n (21) can be factorized then the response of the system to a step input can be written as

$$\bar{y}(s) = \frac{AK(s-z_1)(s-z_2)\cdots(s-z_q)}{s(s-r_1)(s-r_2)\cdots(s-r_p)}$$

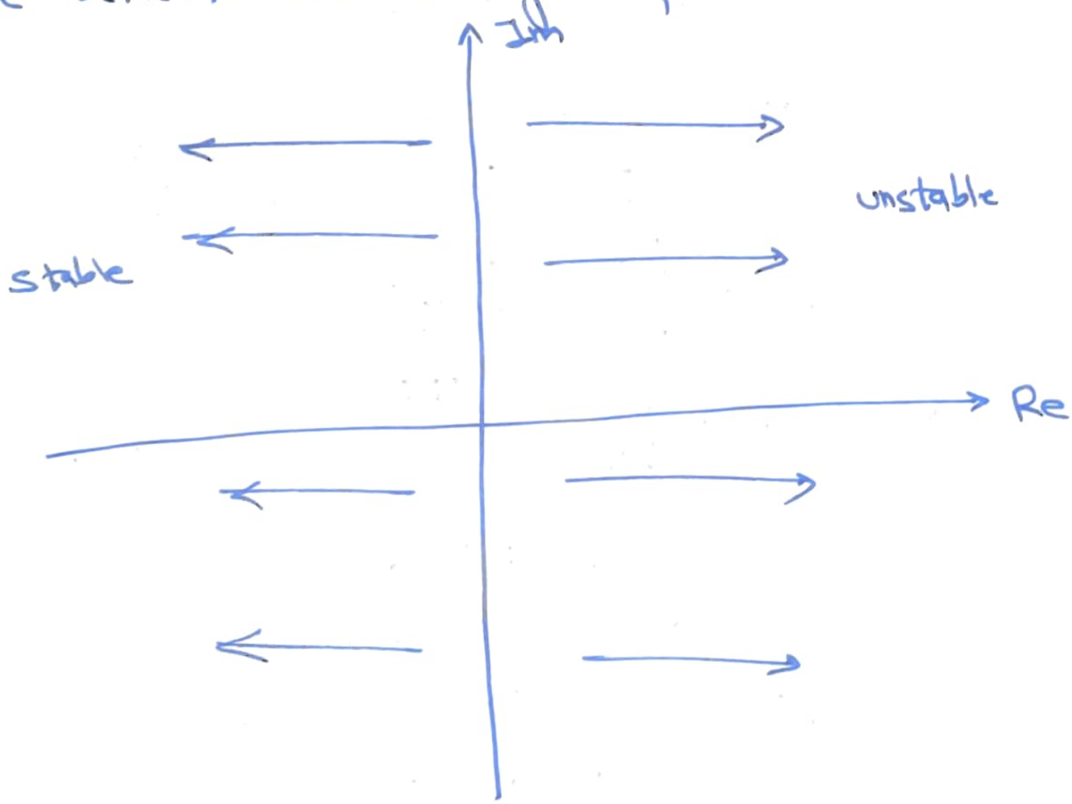
$$\Rightarrow \bar{y}(t) = \frac{A_0}{s} + \sum_{i=1}^{p} \frac{A_i}{s-r_i}$$

$$\Rightarrow y(t) = A_0 + \sum_{i=1}^{p} A_i e^{r_i t} \qquad - (22)$$

By explicitly solving Eq^n (21), we get Eq^n (22) which can be analyzed for stability using the same concepts that were used for analyzing state space systems. We observe r_i's and decide the stability as follows.

- r_i
  - real
    - positive – unstable
    - negative – stable
  - purely imaginary → oscillatory, bound
  - complex
    - real part positive → oscillatory, unbound, unstable
    - real part negative → oscillatory, bound, stable

But r_i in transform domain are the zeros of the transfer function. Hence, the system will be stable if the poles are negative or have negative real part. The zeros do not affect the stability of the system. The same can be reflected on an argand plane as follows:



Frequency-Response Analysis

To understand the dynamics of different systems, we used an ideal step function till now as the input function. Following that method, Laplace transform was used in all the cases. For a sinusoidal input, the following was the response.

$$y(t) = AK\left\{ \frac{\omega\tau}{(\omega\tau)^2+1} e^{-t/\tau} + \frac{1}{\sqrt{(\omega\tau)^2+1}} \sin(\omega t + \phi) \right\} \quad -(1)$$

where $\phi = \tan^{-1}(-\omega\tau)$ $\quad -(2)$

The ideal sine input used in this case was $u(t) = A\sin\omega t$. Eq^n (1) can be seen to consist of a sinusoidal response and a transient response which dies as the exponential of $-t/\tau$. In 4-5 time constants, this part of the response dies down and the response becomes sinusoidal. This part is called the "ultimate periodic response". Hence, both input as well as the output are sinusoidal as $t \to \infty$. However, the amplitude of the response differs from that of the input. Further, there is a phase difference between the input and the output. We can define the quantities amplitude ratio (AR) and phase angle (ϕ) as follows:

$$AR = \frac{AK}{\sqrt{(\omega\tau)^2 + 1}} \quad - (3)$$

$$\phi = \tan^{-1}(\omega\tau) \quad - (4)$$

Hence, only when the input is a sine function then the output is known to us as a sine function. All we need to know to characterize the output is AR and ϕ, both of which can be seen to be functions of ω. This analysis of dependence of AR and ϕ on ω for sinusoidal input-output is called frequency-response analysis.

For solving a system in Laplace domain, we required the Laplace transform domain model of the system. Similarly, to carry out frequency-response analysis, we need frequency-response model which is the model of the system obtained by substituting $s = j\omega$ in the Laplace domain model. j in this model is $\sqrt{-1}$, the imaginary index.

For a general system described by a transfer function $g(s)$ subject to a sinusoidal input,

$$\bar{y}(s) = g(s)\,\bar{u}(s)$$

$$\Rightarrow \bar{y}(s) = g(s) \frac{A\omega}{s^2 + \omega^2} \quad - (5)$$

For a general transfer function, $g(s)$ can be assumed to be a rational polynomial transfer function. Hence, Eqn (5) can be written as the following equation using partial fractions.

$$\bar{y}(s) = \left(\sum_{i=1}^{n} \frac{A_i}{s - r_i} \right) + \frac{B_1}{s - j\omega} + \frac{B_2}{s + j\omega}$$

$$\Rightarrow y(t) = \left(\sum_{i=1}^{n} A_i e^{r_i t} \right) + B_1 e^{j\omega t} + B_2 e^{-j\omega t} \quad - (6)$$

It can be seen from Eqn (6) that only the terms with B_i will survive in an ultimate periodic response in a stable system.

$$\Rightarrow y(t) = B_1 e^{j\omega t} + B_2 e^{-j\omega t} \quad - (7)$$

The constants B_1 and B_2 can be evaluated using the usual method of partial fractions as follows:

$$B_1 = \frac{A\, g(j\omega)}{2j} \; ; \; B_2 = \frac{A\, g(-j\omega)}{-2j}$$

$$\Rightarrow y(t) = \frac{A g(j\omega)}{2j} e^{j\omega t} + \frac{A g(-j\omega)}{-2j} e^{-j\omega t}$$

The conversion of $e^{j\omega t}$ and $e^{-j\omega t}$ to sines and cosines gives the following.

$$y(t) = A\left\{\frac{g(j\omega) - g(-j\omega)}{2j}\right\}\cos\omega t$$

$$+ A\left\{\frac{g(j\omega) - g(-j\omega)}{2}\right\}\sin\omega t \quad -(8)$$

Since $g(j\omega)$ is obtained by substituting $s = j\omega$ in $g(s)$ and $g(s)$ is a rational polynomial function, $g(j\omega)$ is in general a complex variable. Hence, the followings hold true

$$\left.\begin{aligned} g(j\omega) &= Re(\omega) + j\,Im(\omega) \\ g(-j\omega) &= Re(\omega) - j\,Im(\omega) \end{aligned}\right\} \quad -(9)$$

Substitution of Eq^n (9) in Eq^n (8) upon simplification gives the following.

$$y(t) = A\left[Re(\omega)\sin\omega t + Im(\omega)\cos\omega t\right] \quad -(10)$$

The above equation can be simplified using the following trignometric identity.

$$p\sin\theta + q\cos\theta = r\sin(\theta + \phi)$$

where,

$$f = \sqrt{p^2 + q^2}$$

$$\psi = \tan^{-1}(q/p)$$

Hence for the present case, Eqⁿ (10) can be re-written as

$$y(t) = Af \sin(\omega t + \psi) \quad -(11)$$

where,

$$f = \sqrt{[Re(\omega)]^2 + [Im(\omega)]^2}$$

$$\psi = \tan^{-1}\left[\frac{Im(\omega)}{Re(\omega)}\right]$$

Now the input and output of the system are given as

$$u(t) = A \sin \omega t$$

$$y(t) = Af \sin(\omega t + \psi)$$

$$\Rightarrow AR = f = \sqrt{[Im(\omega)]^2 + [Re(\omega)]^2} \quad -(12)$$

Hence the amplitude ratio is obtained by simply determining the modulus of the frequency - domain transfer function $g(j\omega)$.

Further,

$$\psi = \tan^{-1}\left[\frac{Im(\omega)}{Re(\omega)}\right] \quad -(13)$$

Hence, the phase angle of the response is obtained by determining the argument of the frequency-domain transfer function $g(j\omega)$.

We know in general, a complex number can be written in Cartesian or polar form. When $g(j\omega)$ is of the rational form $g(j\omega) = N(j\omega)/D(j\omega)$ then $g(j\omega)$ can be converted to Cartesian form by multiplying and dividing $g(j\omega)$ by complex conjugate of $D(j\omega)$. The AR and ϕ can be obtained in this form using Eqⁿs (12) and (13). Alternatively, $N(j\omega)$ and $D(j\omega)$ can be converted to polar forms.

$$g(j\omega) = \frac{|N(j\omega)|\, e^{\angle N(j\omega)}}{|D(j\omega)|\, e^{\angle D(j\omega)}}$$

AR and ϕ can be obtained from the expression of the above form as follows.

$$AR = \left|\frac{N(j\omega)}{D(j\omega)}\right| \quad -(14)$$

$$\phi = \angle N(j\omega) - \angle D(j\omega) \quad -(15)$$

Now we apply the previously developed procedure to different systems. Let us first consider a first order system. The general transfer function is given as

$$g(s) = \frac{K}{\tau s + 1}$$

$$\Rightarrow g(j\omega) = \frac{K}{j\omega\tau + 1}$$

$$\Rightarrow g(j\omega) = \frac{K}{1 + j\omega\tau} \times \frac{1 - j\omega\tau}{1 - j\omega\tau}$$

$$\Rightarrow g(j\omega) = \frac{K(1 - j\omega\tau)}{1 + \omega^2\tau^2}$$

$$\Rightarrow g(j\omega) = \frac{K}{1+\omega^2\tau^2} - \frac{K\omega\tau}{1+\omega^2\tau^2} j$$

$$\Rightarrow AR = \frac{K}{\sqrt{1+\omega^2\tau^2}} \qquad -(16)$$

$$\phi = -\tan^{-1}(\omega\tau) \qquad -(17)$$

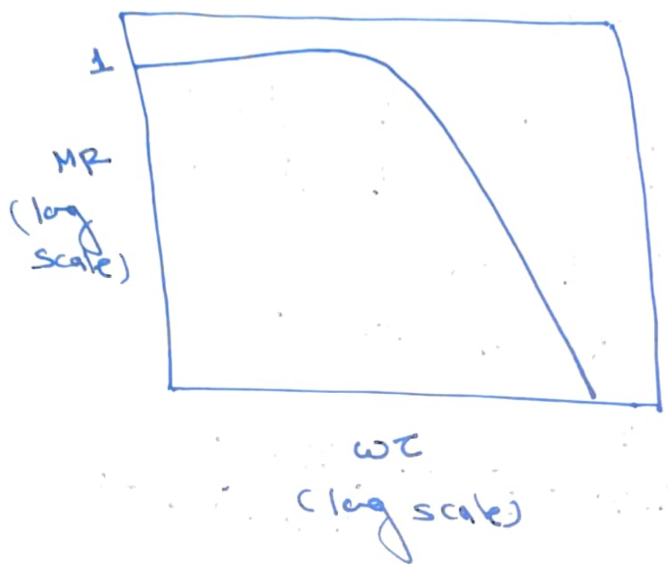
We can see that the same result was obtained previously by solving the Laplace domain model. A useful way of presenting the results is Bode plot which shows the variation of AR with ω and ϕ with ω. The axes are often scaled as follows.

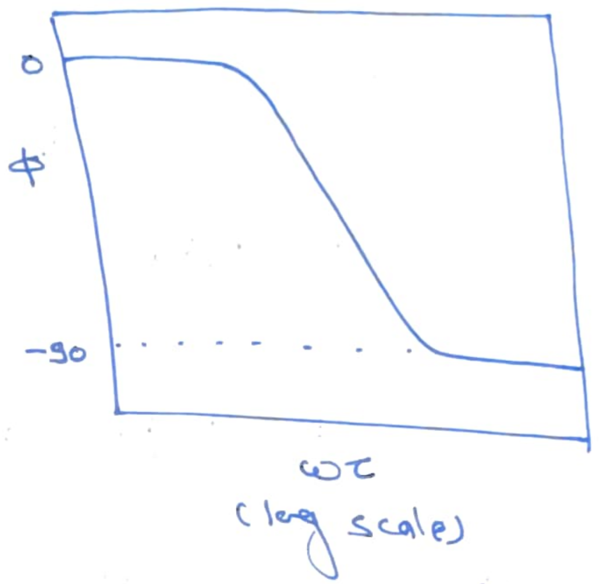
Magnitude ratio MR = AR / K - plotted on log scale

x-axis : $\omega\tau$ - plotted on log scale

ϕ : plotted on natural scale.

Bode plot of frequency-response analysis of a first order system is as shown below.





Now we carry out the analysis for a second order system.

$$g(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$$

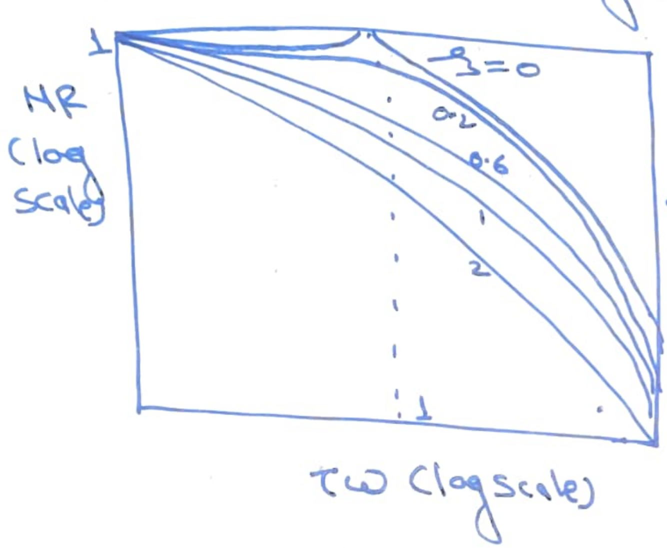
$$\Rightarrow g(j\omega) = \frac{K}{(1-\omega^2\tau^2) + 2\xi\tau\omega j}$$

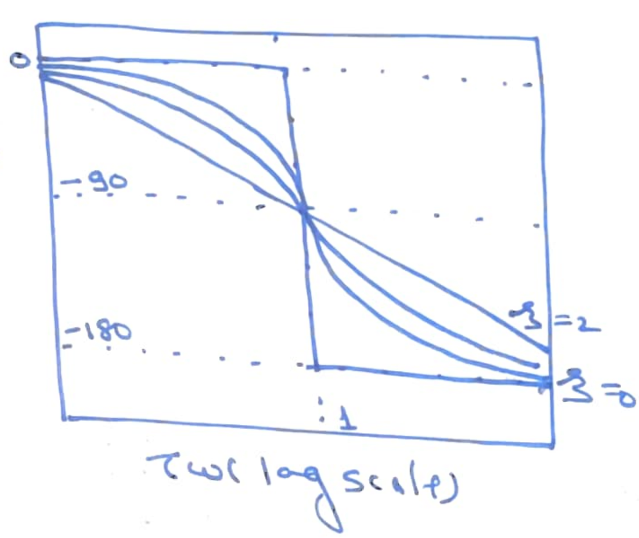
Following the usual conversion to Cartesian form,

$$AR = \frac{K}{\sqrt{(1-\omega^2\tau^2) + (2\xi\tau\omega)^2}} \quad - (18)$$

$$\phi = \tan^{-1}\left[\frac{-2\xi\tau\omega}{1-\omega^2\tau^2}\right] \quad - (19)$$

It can be seen from the above two equations that AR and ϕ are also functions of the parameter ξ. Hence, the Bode plot for a second order system will contain several curves corresponding to different values of ξ. Bode plots for a second order system are shown below.





With a decrease in ξ, the value of AR increases. It goes through a maxima with a change in frequency. This frequency is called resonance frequency.

As seen previously, a second order dynamics is also observed when two first orders are connected in series. In such a case the overall transfer function is given as

$$g(s) = \frac{K_1 K_2}{(\tau_1 s+1)(\tau_2 s+1)}$$

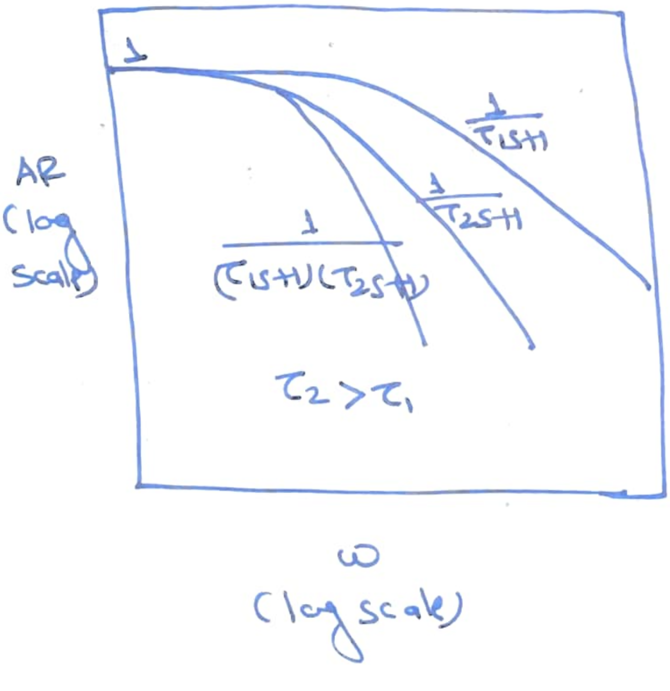
$$\Rightarrow g(j\omega) = \frac{K_1 K_2}{(1+\tau_1\omega j)(1+\tau_2\omega j)} \quad -(20)$$

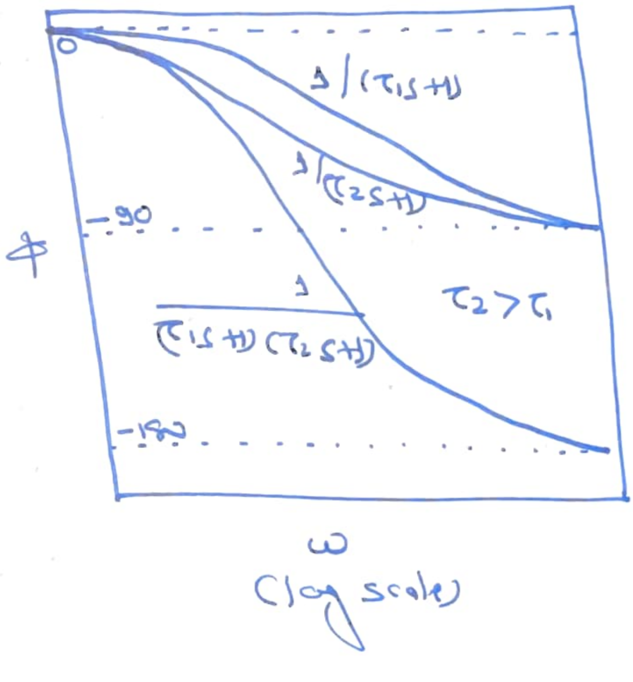
Following the usual procedure, we get

$$AR = K_1 K_2 \frac{1}{\sqrt{1+\tau_1^2\omega^2}} \frac{1}{\sqrt{1+\tau_2^2\omega^2}} \quad -(21)$$

$$\phi = \tan^{-1}(-\omega\tau_1) + \tan^{-1}(-\omega\tau_2) \quad -(22)$$

The Bode diagrams look like below.





Now we generalize the frequency-response analysis for systems in series. If the individual transfer functions are $g_i(s)$ then for N systems in series, the overall transfer function is given as

$$g(s) = g_1(s)\, g_2(s) \ldots g_N(s)$$

$$\Rightarrow g(j\omega) = g_1(j\omega)\, g_2(j\omega) \ldots g_N(j\omega) \qquad -(23)$$

If each individual frequency domain transfer function is expressed in polar form then

$$g_i(j\omega) = |g_i(j\omega)|\, e^{j\phi_i}$$

$$\Rightarrow g(j\omega) = |g(j\omega)|\, e^{j\phi}$$

$$= |g_1(j\omega)|\,|g_2(j\omega)| \ldots |g_N(j\omega)|\, e^{j(\phi_1 + \phi_2 + \ldots \phi_N)}$$

The comparison of LHS with RHS gives the overall AR and overall ϕ as follows.

$$AR_{overall} = AR_1 \times AR_2 \times \ldots \times AR_N \qquad -(24)$$

$$\phi_{overall} = \sum_{i=1}^{N} \phi_i \qquad -(25)$$

On the basis of the foregoing discussion, we can now carry out the frequency-response analysis of all the cases we came across during the transform domain analysis.

Pure gain systems:

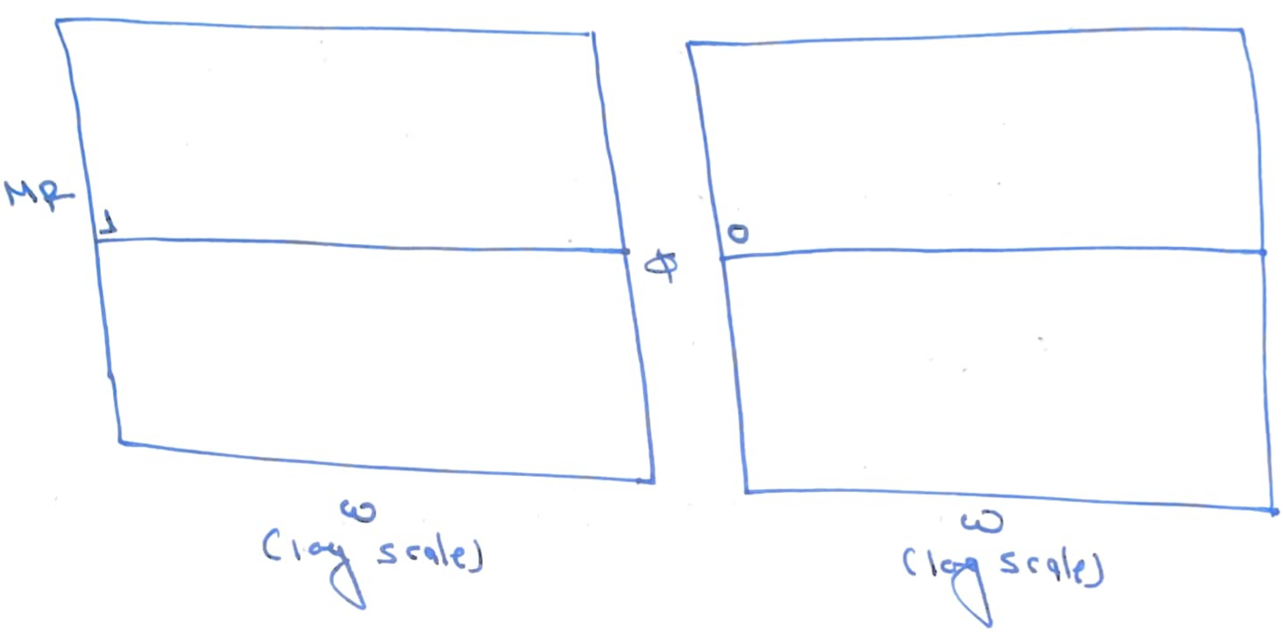
$$g(s) = K$$

$$\Rightarrow g(j\omega) = K = \text{pure real number}$$

$$\Rightarrow AR = K$$

$$\phi = 0$$

Hence, the Bode diagram can be plotted as follows:



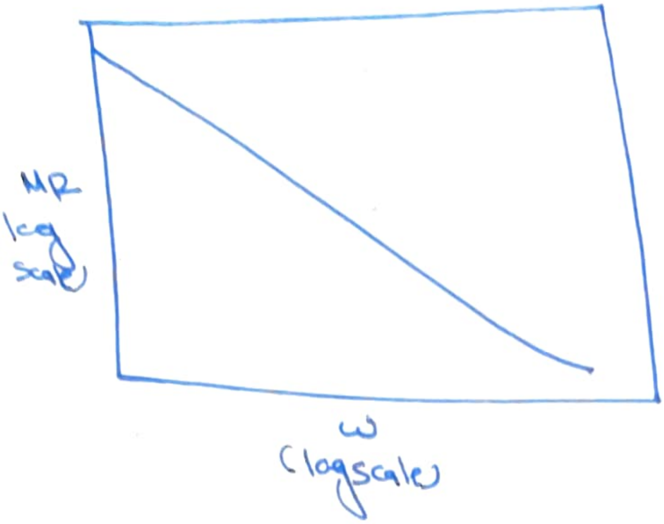
Pure capacity systems:

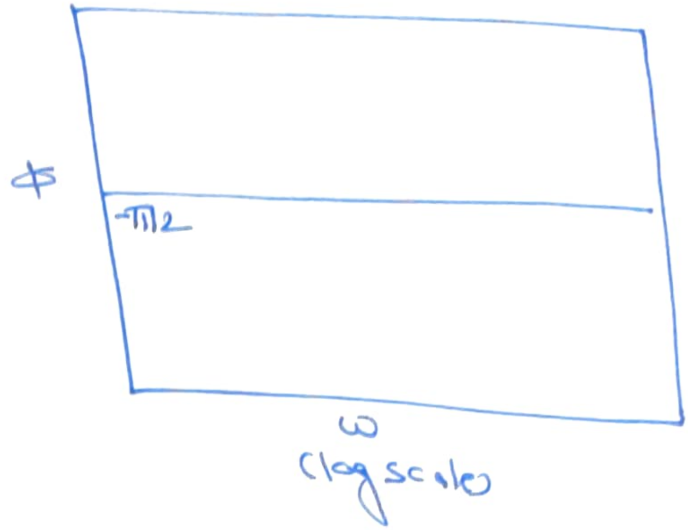
$$g(s) = \frac{K}{s}$$

$$\Rightarrow g(j\omega) = \frac{K}{j\omega}$$

$$\Rightarrow AR = \frac{K}{\omega}$$

$$\phi = -\pi/2$$





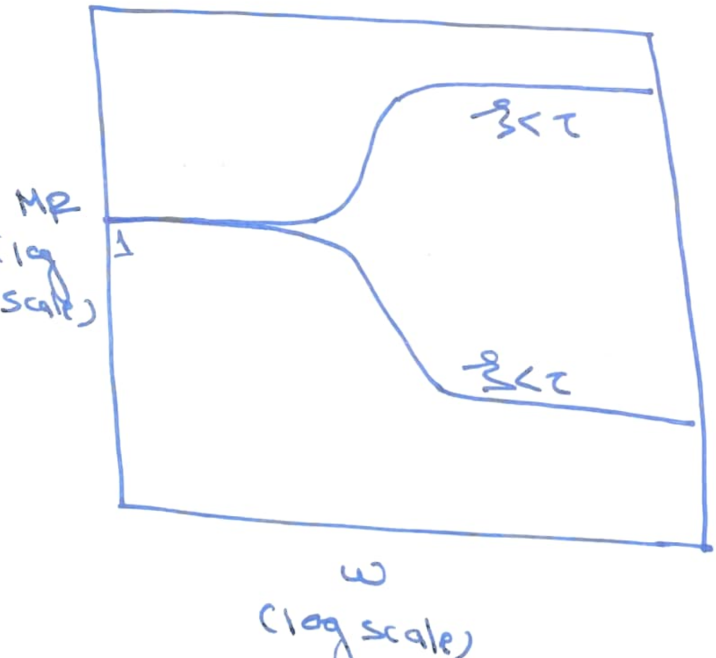
Lead-lag systems:

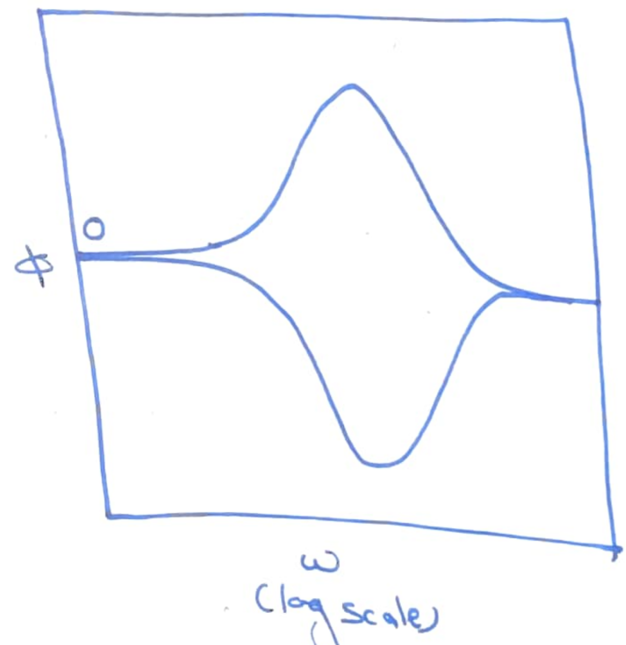
$$g(s) = K\left(\frac{\xi s+1}{\tau s+1}\right)$$

$$\Rightarrow g(j\omega) = K\left(\frac{\xi j\omega+1}{\tau j\omega+1}\right)$$

$$\Rightarrow AR = \frac{K\sqrt{1+\omega^2\xi^2}}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi = \tan^{-1}(\omega\xi) - \tan^{-1}(\omega\tau)$$





We learn a general recepie for obtaining the frequency-response analysis for any system for which the transform domain is available. Firstly, the frequency-response domain model has to be obtained by substituting $j\omega$ for s. Then the modulus and argument of the resulting complex $g(j\omega)$ has to be determined. Using the modulus and argument, AR and ϕ are obtained, respectively. This is followed by plotting Bode plots.

* Do not try to by-heart the Bode plots. Try the following steps.
    - Obtain the expressions for AR and ϕ.
    - Take log scale for AR and linear scale for ϕ. Take log scale for ω.
    - Determine the behaviour and value of the variables close to 1.
    - Determine the asymptotes and draw the plots.
    - Using any plotting software, check whether you get the same trend.

Dynamics of discrete-time systems

Most control systems these days employ digital computers to implement the control algorithm. We know that a digital computer can process only digital signals which are not continuous but discrete. The process variable, on the other hand may be a continuous variable. Hence analog signals from a transducer need to be converted to digital signals. The digital computer will then process the control algorithm and produce digital control action which will be again discrete in time. This, however, may be undesired: opening and closing of a valve repeatedly for example. Hence, the digital control action again needs to be converted to analog signals. Hence, analysis of dynamics of a discrete-time system involves the followings:

(a) Conversion of analog input signals to digital signals

(b) Conversion of continuous-time models to discrete time models

(c) Conversion of discrete-time output signals to analog signals.

We first have a look into the conversion of continuous-time models to discrete-time models. Differential equations in this process are converted to difference equations. Continuous time in this case has to be discretized. If sampling of signals is done at every "T" time units, then the system "hops" from time $t=0$ to $t=T$ and then from $t=T$ to $t=2T$ and so on.

Hence, Δt is a constant in this case which is equal to T. Hence, the first derivative can be approximated as

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{T} \qquad - (1)$$

The above derivative is said to be only an approximation as it can be seen as the truncation of Taylor series expansion.

A general first order system, represented as

$$\frac{dy}{dt} = f(y, a) \qquad -(2)$$

can, therefore, be converted to a discrete-time system as given below.

$$\frac{y_{n+1} - y_n}{T} = f(y_n, a_n)$$

$$\Rightarrow \quad y_{n+1} = y_n + T f(y_n, a_n) \qquad -(3)$$

The difference equation given by Eq^n (3) is the discrete-time model of the continuous-time model of Eq^n (2). In the standard form,

$$\tau \frac{dy}{dt} + y = K u(t)$$

$$\Rightarrow \tau \left(\frac{y_{n+1} - y_n}{T} \right) + y_n = K u_n$$

$$\Rightarrow \quad y_{n+1} = \left(1 - \frac{T}{\tau}\right) y_n + \frac{KT}{\tau} u_n \qquad -(4)$$

It is to be noted that y_n etc are the values of the signal y at time $t = nT$. In all the above "explicit" cases to determine the signal at time $(n+1)T$, the signal and input at nT must be known.

The continuous-time model for a second-order system is

$$\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau\frac{dy}{dt} + y = Ku(t) \qquad -(5)$$

We need the definition of second derivative which can be obtained by using the definition of the first derivative successively.

$$\frac{d^2y}{dt^2} = \frac{1}{T^2}\left(y_{n+2} - 2y_{n+1} + y_n\right) \qquad -(6)$$

$$\Rightarrow \frac{\tau^2}{T^2}\left(y_{n+2} - 2y_{n+1} + y_n\right) + \frac{2\xi\tau}{T}\left(y_{n+1} - y_n\right)$$

$$+ y_n = Ku_n$$

$$\Rightarrow y_{n+2} = 2\left(1 - \frac{\xi T}{\tau}\right) y_{n+1} - \left(\frac{T^2}{\tau^2} - \frac{2\xi T}{\tau}\right) y_n$$

$$+ \frac{KT^2u_n}{\tau^2} \qquad -(7)$$

It can be seen from Eq^n (7) that to determine the value of the signal at the next discrete time instance, the values of the signal at previous two time instances are required.

The same analysis can now be extended to multiple input - multiple output systems. Consider the following case.

$$\frac{dy_1}{dt} + a_{11} y_1 + a_{12} y_2 = b_{11} u_1 + b_{12} u_2$$

$$\frac{dy_2}{dt} + a_{21} y_1 + a_{22} y_2 = b_{21} u_1 + b_{22} u_2$$

$$\Rightarrow \frac{y_{1,n+1} - y_{1,n}}{T} + a_{11} y_{1,n} + a_{12} y_{2,n} = b_{11} u_{1,n} + b_{12} u_{2,n}$$

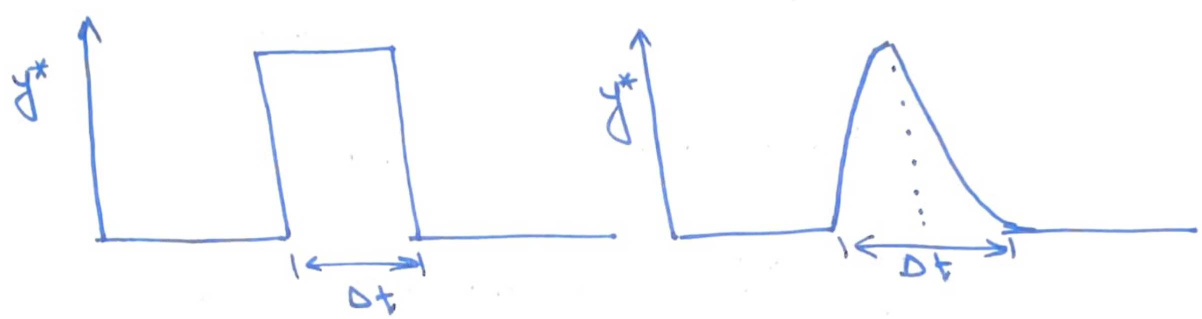
$$\frac{y_{2,n+1} - y_{2,n}}{T} + a_{21} y_{1,n} + a_{22} y_{2,n} = b_{21} u_{1,n} + b_{22} u_{2,n}$$

$$\Rightarrow y_{1,n+1} = (1 - T a_{11}) y_{1,n} - T a_{12} y_{2,n} + T(b_{11} u_{1,n} + b_{12} u_{2,n})$$

$$y_{2,n+1} = -T a_{21} y_{1,n} + (1 - T a_{22}) y_{2,n} + T(b_{21} u_{1,n} + b_{22} u_{2,n})$$

Using the above principles it is possible to obtain the discrete-time model of any continuous process.

Now we look at the procedure of converting a continuous signal to a discrete signal. Physically the conversion is accomplished by placing a switch which takes the input signal after a time interval T. Hence, the signal on switching the switch on rises to the value of the signal and then decays to zero. An output from an ideal and a real sampler would look like the one shown below.



Ideal real

If the sampler acts instantaneously then $\Delta t \to 0$ and the area under the "impulse" equals the strength of the signal. Hence, the system can be modelled as follows.

$$y^*(nT) = y(nT)\,\delta(t - nT) \quad - (8)$$

where y^* is the discrete signal at $t = nT$ and δ is the impulse or Dirac function.

The continuous signal can hence be seen to have converted to discrete signal expressed by the following series.

$$y^*(t) = y^*(0) + y^*(T) + y^*(2T) + \dots$$
$$= y(0)\delta(t) + y(T)\delta(t-T) + y(2T)\delta(t-2T) + \dots$$

$$\Rightarrow y^*(t) = \sum_{n=0}^{\infty} y(nT)\,\delta(t-nT) \qquad -(9)$$

The above equation gives the discrete signal from a continuous signal. In transform domain, the model is written as follows.

$$\bar{y}^*(s) = \sum_{n=0}^{\infty} y(nT)\,\mathcal{L}[\delta(t-nT)]$$

$$\Rightarrow \bar{y}^*(s) = \sum_{n=0}^{\infty} y(nT)\,e^{-nTs} \qquad -(10)$$

Now we look at the procedure of converting a discrete signal to a continuous signal. Consider a set of signals $m(T)$, $m(2T)$, $m(3T)$ ... and it is desired to obtain a continuous signal from it. Let the continuous signal be $m(t)$. We do a Taylor series expansion of $m(T)$ about a sampled value of $m(nT)$.

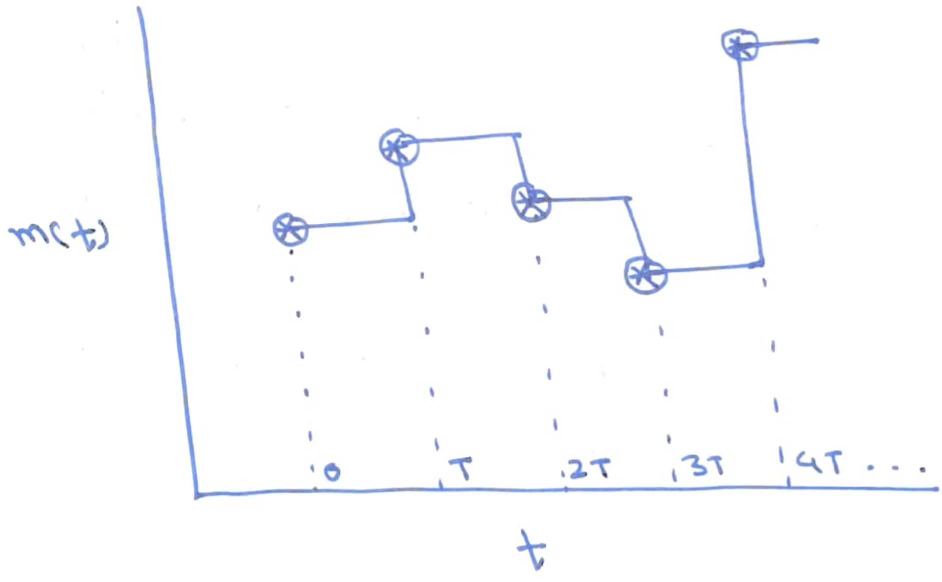
$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)\Big|_{t=nT}(t-nT) + \frac{1}{2}\left(\frac{d^2m}{dt^2}\right)\Big|_{t=nT}(t-nT)^2 + \cdots \quad -(11)$$

From the above expansion, we get different "hold elements" of different orders based on truncation.

Zero-order hold element is obtained when we take only the zero-order term.

$$m(t) \approx m(nT), \quad nT \leq t < (n+1)T \quad -(12)$$

The RHS of Eq^n (12) is a constant. Hence, the output of a zero-order hold is a pulse function having a constant height of $m(nT)$ and duration T. Hence, the continuous signal would look like a collection of step functions as shown below.



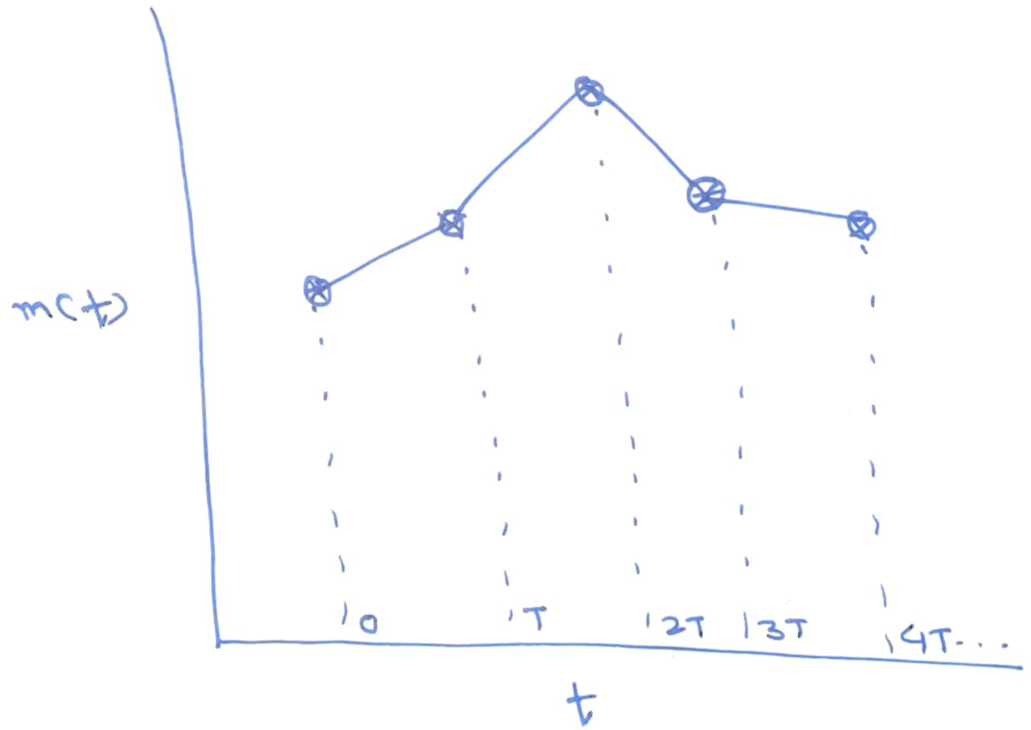
The first-order hold element is obtained by considering terms upto the first derivative and discretizing the derivative.

$$m(t) = m(nT) + \left(\frac{dm}{dt}\right)\Big|_{t=nT} (t-nT)$$

$$\left(\frac{dm}{dt}\right)\Big|_{t=nT} = \frac{m(nT) - m[(n-1)T]}{T}$$

$$\Rightarrow m(t) = m(nT) + \frac{m(nT) - m[(n-1)T]}{T}(t-nT) \qquad -(13)$$

It can be seen that to produce a continuous signal from first-order hold, two previous signals are required. This is in contrast to zero-order hold which requires only one signal. The following nature of continuous signal from first order hold can be easily deduced.



The first-order hold tends to produce continuous signal having magnitude which can be higher than the actual. Hence, large control actions are often enacted in such a case.

For a zeroth-order hold,

$$m(t) \approx m(nT), \quad nT \leq t < (n+1)T$$

$$\Rightarrow \overline{m}(s) = m(nT)\left(\frac{1-e^{-sT}}{s}\right) \quad -(14)$$

where the Laplace transform of a pulse function has been invoked. Hence the transfer function for a zeroth-order hold is given as

$$g_0(s) = \frac{1-e^{-sT}}{s} \quad -(15)$$

Similarly for a first-order hold,

$$g_1(s) = \frac{1+sT}{T}\left(\frac{1-e^{-sT}}{s}\right)^2 \quad -(16)$$

Till now we made use of Laplace transform to analyze the problem. For handling discrete systems, it is more convenient to use Z-transforms which we now develop.

Z-transforms: Z transform takes a discrete-time signal from time domain to z domain. For continuous or discrete signals, z-transform is defined as follows:

$$Z[y(t)] = \bar{Y}(z) = \sum_{n=0}^{\infty} y(nT)\, z^{-n} \qquad -(17)$$

$$Z\{y(0), y(T), y(2T) \ldots\} = \bar{Y}(z)$$

$$= \sum_{n=0}^{\infty} y(nT)\, z^{-n} \qquad -(18)$$

If $y(t)$ is a continuous signal and $y^*(t)$ is a sequence of discrete-time signals produced by the sampler then we saw previously

$$\bar{Y}^*(s) = \sum_{n=0}^{\infty} y(nT)\, e^{-nTs}$$

when $e^{Ts} = z$ then the above equation becomes

$$\bar{Y}^*(s) = \sum_{n=0}^{\infty} y(nT)\, z^{-n}$$

$$\Rightarrow \bar{Y}^*(s) = \bar{Y}(z) \qquad -(19)$$

Hence, the z-transform of a discrete-time signal is a special case of its Laplace transform when s and z are related as $z = e^{Ts}$.

Now we look at z-transforms of some basic forcing functions.

Unit step function:

Unit step function $= \{1, 1, 1, \ldots\}$

$\Rightarrow$ Z (unit step function)

$$= 1 \cdot z^{0} + 1 \cdot z^{-1} + 1 \cdot z^{-2} + \ldots$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1}{1 - z^{-1}}$$

Ramp function:

Ramp function $= \{0, aT, 2aT, 3aT, \ldots\}$

$\Rightarrow$ Z (Ramp function)

$$= 0 \cdot z^{0} + aT \cdot z^{-1} + 2aT \cdot z^{-2} + \ldots$$

$$= aT\,(1 + 2z^{-1} + 3z^{-2} + \ldots)$$

$$= \frac{aT z^{-1}}{(1 - z^{-1})^{2}}$$

Exponential function

$$Z(e^{-at}) = \sum_{n=0}^{\infty} e^{-anT} z^{-n}$$

$$= \frac{1}{1 - e^{aT} z^{-1}}$$

$$= \frac{z}{z - e^{aT}}$$

Following this method, z-transform of any function can be derived. We provide the table of z-transform below and leave their derivation as an exercise.

1. Unit impulse $\delta(t) = 1 \rightarrow 1$

2. Unit step $\rightarrow \dfrac{1}{1 - z^{-1}}$

3. $f(t) = at \rightarrow \dfrac{aT z^{-1}}{(1 - z^{-1})^2}$

4. $f(t) = t^n \rightarrow \lim\limits_{a \to 0} (-1)^n \dfrac{\partial^n}{\partial a^n} \left(\dfrac{1}{1 - e^{-aT} z^{-1}} \right)$

5. $f(t) = e^{-at} \rightarrow \dfrac{1}{1 - e^{-aT} z^{-1}}$

6. $f(t) = te^{-at} \rightarrow \dfrac{Te^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^2}$

7. $f(t) = \sin\omega t \rightarrow \dfrac{z^{-1} \sin\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$

8. $f(t) = \cos\omega t \rightarrow \dfrac{1 - z^{-1}\cos\omega T}{1 - 2z^{-1}\cos\omega T + z^{-2}}$

9. $f(t) = e^{-at}\sin\omega t \rightarrow \dfrac{z^{-1} e^{-aT} \sin\omega T}{1 - 2z^{-1} e^{-aT}\cos\omega T + e^{-2aT} z^{-2}}$

10. $f(t) = e^{-at}\cos\omega t \rightarrow \dfrac{1 - z^{-1} e^{-aT} \cos\omega T}{1 - 2z^{-1} e^{-aT}\cos\omega T + e^{-2aT} z^{-2}}$

We give other properties of Z-transforms without proof.

Translated functions : If a function $f(t)$ is delayed by t_d such that $t_d = kT$, $k \in \mathbb{I}^+$, then

$$Z[f(t-t_d)] = \bar{f}(z)\, z^{-k} \qquad -(20)$$

If the signal is advanced by $t_a = kT$ then

$$Z[f(t+t_a)] = \bar{f}(z)\, z^{k} \qquad -(21)$$

Initial value theorem : According to this theorem,

$$\lim_{t \to 0} [y(t)] = \lim_{z \to \infty} [\bar{y}(z)] \qquad -(21)$$

Final value theorem : According to this theorem,

$$\lim_{t \to \infty} [y(t)] = \lim_{z \to 1} [(1 - z^{-1})\, \bar{y}(z)] \qquad -(22)$$

Transform of a derivative : Using first order approximation,

$$Z\left(\frac{df}{dt}\right) = \bar{y}(z) = \frac{1}{T}(1 - z^{-1})\, \bar{f}(z) \qquad -(23)$$

Using second order approximation,

$$Z\left(\frac{df}{dt}\right) = \bar{y}(z) = \frac{1}{2T}(1 - z^{-2})\, \bar{f}(z) \qquad -(24)$$

Analysis of first order dynamics

For a first order system, the discrete-time model was derived previously (Eq^n 4) as

$$y_{n+1} = \left(1 - \frac{T}{\tau}\right) y_n + \frac{KT}{\tau} u_n$$

Clubbing various constants, we can write the above equation as

$$y_{n+1} + a y_n = b u_n \qquad -(25)$$

$$\Rightarrow y_{n+1} = -a y_n + b u_n \qquad -(26)$$

For a unit step input

$$u_n = 1 \quad , \quad n = 0, 1, 2, \ldots$$

Replacing $-a$ by a for notational convenience,

$$\Rightarrow y_{n+1} = a y_n + b \qquad -(27)$$

$$y_1 = a y_0 + b$$

$$\Rightarrow y_2 = a y_1 + b$$

$$= a(a y_0 + b) + b$$

$$= a^2 y_0 + ab + b$$

The general expression for y_n is

$$y_n = a^n y_0 + b\left(a^{n-1} + a^{n-2} + \cdots + 1\right) \qquad -(28)$$

If y in the present case is the deviation variable then $y_0 = 0$.

$$\Rightarrow \quad y_n = b(a^{n-1} + a^{n-2} + \dots + 1)$$

For $a \neq 1$, the geometric series above gives the following result.

$$y_n = b\left(\frac{1-a^n}{1-a}\right) \qquad -(29)$$

Eq^n (29) gives the response of a discrete-time system subject to a unit input. It can be seen that the system is bounded (stable) for $|a| < 1$. The steady-state value can be obtained as $y(\infty) = b/(1-a)$. The system will exhibit unbounded response (unstable) for $|a| > 1$. This is similar to the case of continuous-time model. The response is sketched below.

Now we carry out the previous analysis using Z-transform. Like always, the Z-domain transfer function, by definition, is

$$g(z) = \frac{\bar{y}(z)}{\bar{u}(z)}$$

As seen previously, for a unit step function,

$$\bar{u}(z) = \frac{1}{1 - z^{-1}}$$

We need to determine the Z-domain transfer function. For this we use the following theorem without proof.

The pulse transfer function, $g(z)$, of a sampled-data process without a hold is the z-transform of inverse Laplace transform, $g(t)$, of the continuous process.

Following the above, if Laplace transform $g(s)$ is available then one first needs to obtain the inverse Laplace transform of $g(s)$, i.e. $g(t)$. From $g(t)$, obtain the sequence $g(n)$. The Z-transform of this sequence is the required transform.

Now we apply this procedure for a first order system for which

$$g(s) = \frac{K}{\tau s + 1}$$

$$\Rightarrow g(t) = \frac{K}{\tau} e^{-t/\tau}$$

$$\Rightarrow g(z) = \sum_{n=0}^{\infty} \frac{K}{\tau} e^{-n\Delta t/\tau} z^{-n}$$

$$\Rightarrow g(z) = \frac{K}{\tau}\left(\frac{1}{1 - a z^{-1}}\right)$$

where $a = e^{-\Delta t/\tau}$

In the present case the Z-domain transfer function can be seen as

$$g(z) = \frac{b z^{-1}}{1 - a z^{-1}}$$

$$\Rightarrow y(z) = \left(\frac{b z^{-1}}{1 - a z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

It is left as an exercise to show using partial fractions,

$$y(n) = b\left[\frac{1}{1-a} - a^n\left(\frac{1}{1-a}\right)\right]$$

$$\Rightarrow y(n) = b\left(\frac{1-a^n}{1-a}\right)$$

as obtained before.

Now we extend the previous discussion to a general linear discrete-time process given as

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \dots a_k y(n-k)$$
$$= b_0 u(n) + b_1 u(n-1) + b_2 u(n-2) + \dots + bm(n-m) \quad -(30)$$

The z-domain transfer function for the above equation can be derived as

$$g(z) = \left(\frac{b_0 + b_1 z^{-1} + b_2 b z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_k z^{-k}}\right) - (30)$$

$$\Rightarrow \bar{y}(z) = \left(\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_k z^{-k}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

Like the analysis that we did for Laplace transform, it is fair to assume that the denominator ~~that~~ has k distinct roots and we can do partial fraction of the above expression of the form

$$\bar{y}(z) = \left(\frac{R_0}{1 - z^{-1}}\right) + \left(\frac{R_1}{1 - p_1 z^{-1}}\right) + \left(\frac{R_2}{1 - p_2 z^{-1}}\right) + \dots + \left(\frac{R_k}{1 - p_k z^{-1}}\right)$$

$$\Rightarrow y(n) = R_0 + R_1 (p_1)^n + R_2 (p_2)^n + \dots + R_k (P_k)^n \quad - (31)$$

For a stable system, $|p_i| < 1$ and hence, the steady-state value $y(\infty) = R_0$. Hence the stability of the system is governed by the sign of roots of denominator of $g(z)$.

In case of a multiple-input multiple-output discrete-time system, $y(n)$ is replaced by the vector $\underline{y}(n)$ and $u(n)$ is replaced by the vector $\underline{u}(n)$. Consider a 2×2 system.

$$y_1(n) + a_{11} y_1(n-1) + a_{12} y_2(n-1) = b_{11} u_1(n-1) + b_{12} u_2(n-1)$$

$$y_2(n) + a_{21} y_1(n-1) + a_{22} y_2(n-1) = b_{21} u_1(n-1) + b_{22} u_2(n-1)$$

We can define the following matrices and vectors.

$$\underline{\underline{\Phi}} = \begin{bmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{bmatrix} ; \quad \underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\underline{y}(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} ; \quad \underline{u}(n) = \begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix}$$

$$\Rightarrow \underline{y}(n) = \underline{\underline{\Phi}}\, \underline{y}(n-1) + \underline{\underline{B}}\, \underline{u}(n-1)$$

The eigenvalues of Φ are indicative of the stability of the system. The system will be stable if every eigenvalue of $|\lambda(\underline{\underline{\Phi}})| < 1$.