Planarity

Polygonal Paths

- A *polygonal path* or *polygonal curve* in the plane is the union of finitely many line segments such that each segment starts at the end of the previous one and no point appears in more than one segment except for common endpoints of consecutive segments.

- In a *polygonal u,v-path*, the beginning of the first segment is *u* and the end of the last segment is *v*.

Drawing of a Graph

- A *drawing* of a graph G is a function that maps each vertex $v \in V(G)$ to a point f(v) in the plane and each edge *uv* to a polygonal f(u),f(v)-path in the plane.
  – The images of vertices are distinct
  – A point in $f(e) \cap f(e')$ other than a common end is a *crossing.*

Planar Graph

- A graph is *planar* if it has a drawing without crossings.
- A *plane graph* is a particular drawing of a planar graph in the plane with no crossings.

Faces

- An *open set* in the plane is a set $U \in \Re^2$ such that for every $p \in U$, all points within some small distance from p belong to U.

- A *region* is an open set U that contains a polygonal u,v-path for every pair $u,v \in U$.

- The *faces* of a plane graph are the maximal regions of the plane that are disjoint from the drawing.

- The length of a face in a plane graph G is the length of the walk in G that bounds it.

Dual Graphs

- Suppose G is a plane graph. The *dual graph* G^* of G is a plane graph having a vertex for each region in G. The edges of G^* correspond to the edges of G as follows:
  - If e is an edge of G that has region X on one side and region Y on the other side, then the corresponding dual edge $e^* \in E(G^*)$ is an edge joining the vertices x, y of G^* that correspond to the faces X,Y of G.

Results

- If $l(F_i)$ denotes the length of face F_i in a plane graph G, then $2e(G) = \Sigma\, l(F_i)$.

- The following statements are equivalent for a plane graph G:
  1. G is bipartite.
  2. Every face of G has even length.
  3. The dual graph G* is Eulerian.

Euler's Formula & other results

- [Euler's Formula:] If a connected plane graph G has n vertices, e edges and f faces, then $n - e + f = 2$

- If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$.
  - If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

More results

- If E is the edge set of a face in some planar embedding of G, then G has an embedding in which E is the edge set of the unbounded face.

- Every minimal non-planar graph is 2-connected.

- Suppose $S = \{x,y\}$ is a 2-cut of G and $G_!,G_2$ are sub-graphs of G such that $G_1 \cup G_2 = G$ and $V(G_1) \cap V(G_2) = S$. Let $H_i = G_i \cup xy$. If G is non-planar, then at least one of H_1,H_2 is non-planar.

Preparation for Kuratowski's Theorem

- Suppose G is a non-planar graph with no Kuratowski subgraph, and G has the fewest edges among such graphs. Then G is 3-connected.
- A 3-connected graph with at least five vertices contains an edge whose contraction leaves a 3-connected graph.
- If G.e has a Kuratowski subgraph, then G also has a Kuratowski subgraph.

Kuratowski's Theorem / Tutte's Theorem

- *Subdividing* and edge means replacing the edge with a path of length 2. Kuratowski proved that G is planar iff G contains no sub-division of K_5 or $K_{3,3}$.

- [Tutte's Theorem:] If G is a 3-connected graph with no subdivision of K_5 or $K_{3,3}$, then G has a convex embedding in the plane. This is a stronger result that Kuratowski's theorem.