Graph Theory: Introduction

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Resources

- Copies of slides available at:

  http://www.facweb.iitkgp.ernet.in/~pallab

- Book to be followed mainly:

  Introduction to Graph Theory

  -- Douglas B West

Graph Theory

- A graph is a discrete structure
  - Mathematically, a relation
- Graph theory is about studying
  - Properties of various types of Graphs
  - … and graph algorithms

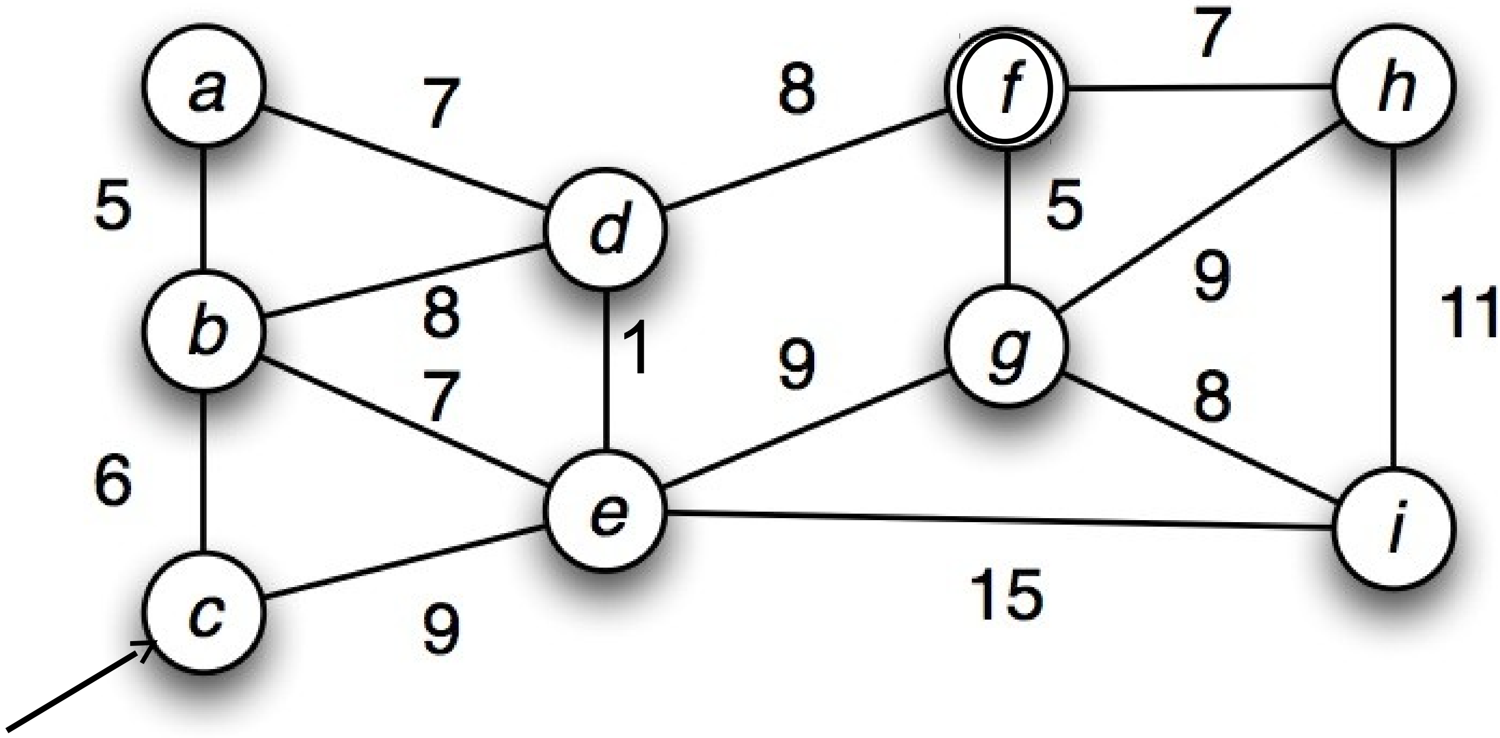
Why should CSE students study graph theory?

Graphs can be used to model problems

- The following table illustrates a number of possible duties for the drivers of a bus company.
- We wish to ensure at the lowest possible cost, that at least one driver is on duty for each hour of the planning period (9 AM to 5 PM).

Duty hours	9 – 1	9 – 11	12 – 3	12 – 5	2 – 5	1 – 4	4 – 5
Cost	300	180	210	380	200	340	90

Graphs can be used to model problems



Graph

- A graph G = (V,E) with *n* vertices and *m* edges consists of:
  - a *vertex set* $V(G) = \{v_1, \ldots, v_n\}$, and
  - an *edge set* $E(G) = \{e_1, \ldots, e_m\}$, where each edge consists of two (possibly equal) vertices called its *endpoints.*
- We write *uv* for an edge *e={u,v}*, and say that *u* and *v* are adjacent
- A *simple graph* is a graph having no loops or multiple edges
  - What is a *loop* ?

Digraph

- A *directed graph* or *digraph* G consists of a vertex set V(G) and an edge set E(G), where each edge is an ordered pair of vertices.
  - A *simple digraph* is a digraph in which each ordered pair of vertices occurs at most once as an edge.
  - Throughout this course we shall consider undirected simple graphs, unless mentioned otherwise.

Complement

- The *complement* G′ of a simple graph G is the simple graph with vertex set V(G) and edge set defined by:
  - $uv \in E(G')$ if and only if $uv \notin E(G)$

Subgraph

- A *subgraph* of a graph G is a graph H, such that:
  - $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
- An *induced subgraph* of G is a subgraph H of G such that E(H) consists of all edges of G whose endpoints belong to V(H)

Complete Graph / Clique

- A *complete graph* or a *clique* is a simple graph in which every pair of vertices is an edge.
  - We use the notation K_n to denote a clique of *n* vertices
  - The complement K_n' of K_n has no edges
  - How does an induced subgraph of a clique look like?

Independent set

- An *independent subset* in a graph G is a vertex subset $S \subseteq V(G)$ that contains no edge of G

Bipartite Graph

- A graph G is *bipartite* if V(G) is the union of two disjoint sets such that each edge of G consists of one vertex from each set.
  - A complete bipartite graph is a bipartite graph whose edge set consists of all pairs having a vertex from each of the two disjoint sets of vertices
  - A complete bipartite graph with partite sets of sizes *r* and *s* is denoted by $K_{r,s}$

K-partite Graph

- A graph G is *k-partite* if V(G) is the union of *k* independent sets.

Chromatic number

- A graph is *k-colorable*, if we can color the vertices of the graph using *k* colors such that the endpoints of each edge have different colors
  - The *chromatic number*, $\chi(G)$ of a graph G is the minimum number of colors required to color G.

Planar Graph

- A graph is *planar* if it can be drawn in the plane without edge crossings

Path & Cycle

- A *path* in a graph is a single vertex or an ordered list of distinct vertices $v_1, \ldots, v_k$ such that $v_{i-1}v_1$ is an edge for all $2 \leq i \leq k$.
  - the ordered list is a *cycle* if $v_k v_1$ is also an edge
  - A path is an *u,v-path* if *u* and *v* are respectively the first and last vertices on the path
  - A path of *n* vertices is denoted by P_n, and a cycle of *n* vertices is denoted by C_n.

Connected Graph

- A graph G is *connected* if it has a *u,v-path* for each pair $u,v \in V(G)$.

Walk and Trail

- A *walk* of length k is a sequence, $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for all i.
- A *trail* is a walk with no repeated edge.
  - A *path* is a walk with no repeated vertex
  - A walk is *closed* if it has length at least one and its endpoints are equal
  - A *cycle* is a closed trail in which “first = last” is the only vertex repetition
  - A *loop* is a cycle of length one

Equivalence Relation

- A *relation* R on a set S is a collection of ordered pairs from S.
- An *equivalence relation* is a relation R that is reflexive, symmetric and transitive.

Graphs as Relations

- A graph is an adjacency relation. For simple undirected graphs the relation is symmetric, and not reflexive.
  - The adjacency relation is not necessarily an equivalence relation, since it is not necessarily transitive.

Graph Isomorphism

- An *isomorphism* from G to H is a bijection $f: V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.
  - We say that *G is isomorphic to H*, written as $G \cong H$, if there is an isomorphism from G to H.
  - Is isomorphism an equivalence relation?

Automorphism

- An *automorphism* of G is a permutation of V(G) that is an isomorphism from G to G.
  - A graph is called *vertex transitive* if for every pair $u,v \in V(G)$ there is an automorphism that maps u to v.

Union, Sum, Join

- The *union* of graphs G and H, written as $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.
  - To specify the *disjoint union* $V(G) \cap V(H) = \phi$, we write G+H.
  - *mG* denotes the graph consisting of *m* pairwise disjoint copies of *G*.
  - The *join* of G and H, written as $G \vee H$ is obtained from G+H by adding the edges
    $\{xy : x \in V(G), y \in V(H)\}$

  Is $(G+H)' = G' \vee H'$?

Cut-vertex, Cut-edge

- The *components* of a graph G are its maximal connected subgraphs.
  - A component is *non-trivial* if it contains an edge.
  - A *cut-edge* or *cut-vertex* of a graph is an edge or vertex whose deletion increases the number of components