Some Counting Problems

Degree

- The *degree* of a vertex v in a graph G, written as $d_G(v)$ or simply $d(v)$, is the number of non-loop edges containing v plus twice the number of loops containing v.
  - The maximum degree is $\Delta(G)$
  - The minimum degree is $\delta(G)$
  - A graph is *regular* if $\Delta(G) = \delta(G)$
  - A graph is *k-regular* if $\Delta(G) = \delta(G) = k$
  - A vertex of degree k is *k-valent*.

Order and Size

- The *order* of a graph G, written as *n(G)*, is the number of vertices in G.
- The *size* of a graph G, written as *e(G)*, is the number of edges in G.

Countings and Bijections

- If G is a graph, then $\sum_{v \in V(G)} d(v) = 2e(G)$

- Every graph has an even number of vertices of odd degree.
  - No graph of odd order is regular with odd degree

- A k-regular graph with *n* vertices has *nk/2* edges.

Countings and Bijections

- Suppose J, H, G are graphs with $J \subseteq H \subseteq G$, and suppose H contains q copies of J. If G contains m copies of J, and the i^{th} copy of J appears in k_i copies of H in G, then G contains $\Sigma_{i=1,m}\, k_i / q$ copies of H.

- For n≥1, there are $2^{\binom{n-1}{2}}$ simple graphs with vertex set $\{v_1, \ldots, v_n\}$ such that every vertex degree is even.

Pigeon-hole Principle

If a set consisting of more than kn objects is partitioned into n classes, then some class receives more than k objects.

- Every simple graph with at least two vertices has two vertices of equal degree.
- If G is a simple n-vertex graph with $\delta(G) \geq (n-1)/2$, then G is connected

Turan Graph

- The *Turan graph* $T_{n,r}$ is the complete r-partite graph with n vertices that has b parts of size $a+1$ and $r-b$ parts of size a, where $a = n/r$ and $b = n-ra$.

- Turan showed that $T_{n,r}$ is the unique largest simple n-vertex graph with no r+1-clique.

Mantel's Theorem & Turan's Theorem

- [Mantel 1907] The maximum number of edges in an n-vertex triangle-free graph is: $\lfloor n^2/4 \rfloor$
- [Turan 1941] Among the n-vertex simple graphs with no r+1-clique, $T_{n,r}$ has the maximum number of edges.