Graph Theory: Cuts and Connectivity

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Vertex Cut and Connectivity

- A ***separating set*** or ***vertex cut*** of a graph G is a set $S \subseteq V(G)$ such that G–S has more than one component or has one remaining vertex.
  - A graph G is ***k-connected*** if every vertex cut has at least *k* vertices.
  - The ***connectivity*** of G, written as $\kappa(G)$, is the minimum size of a vertex cut.

Edge-connectivity

- A ***disconnecting set*** of edges is a set $F \subseteq E(G)$ such that G – F has more than one component.
  - A graph is ***k-edge-connected*** if every disconnecting set has at least *k* edges.
  - The ***edge connectivity*** of G, written as $\kappa'(G)$, is the minimum size of a disconnecting set.

Edge Cut

- Given $S,T \subseteq V(G)$, we write [S,T] for the set of edges having one endpoint in S and the other in T.
  - An *edge cut* is an edge set of the form [S,S′], where S is a nonempty proper subset of V(G).

Results

- $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

- **If S is a subset of the vertices of a graph G, then:**

$$|[S,S']| = [\Sigma_{v\in S}\, d(v)] - 2e(G[S])$$

- **If G is a simple graph and $|[S,S']| < \delta(G)$ for some nonempty proper subset S of V(G), then $|S| > \delta(G)$.**

More results…

- A graph G having at least three vertices is 2-connected if and only if each pair $u,v \in V(G)$ is connected by a pair of internally disjoint *u,v*-paths in G.

- If G is a k-connected graph, and G′ is obtained from G by adding a new vertex *y* adjacent to at least *k* vertices in G, then G′ is k-connected.

And more …

If $n(G) \geq 3$, then the following conditions are equivalent (and characterize 2-connected graphs)

(A) G is connected and has no cut vertex.

(B) For all $x,y \in V(G)$, there are internally disjoint x,y-paths

(C) For all $x,y \in V(G)$, there is a cycle through x and y.

(D) $\delta(G) \geq 1$, and every pair of edges in G lies on a common cycle

x,y-separator

- Given $x,y \in V(G)$, a set $S \subseteq V(G) - \{x,y\}$ is an ***x,y-separator*** or a ***x,y-cut*** if $G-S$ has no *x,y*-path.
  - Let $\kappa(x,y)$ be the minimum size of an *x,y*-cut.
- Let $\lambda(x,y)$ be the minimum size of a set of ***pair-wise internally disjoint x,y-paths***.
  - Let $\lambda(G)$ be the largest k such that $\lambda(x,y) \geq k$ for all $x,y \in V(G)$.
  - For $X,Y \subseteq V(G)$, an X,Y-*path* is a path having first vertex in *X*, last vertex in *Y*, and no other vertex in $X \cup Y$.

Menger's Theorem

- If *x,y* are vertices of a graph G and $x,y \notin E(G)$, then the **minimum size of an *x,y*-cut** equals the **maximum number of pair-wise internally disjoint *x,y*-paths**.

- [Corollary] The connectivity of G equals the maximum *k* such that $\lambda(x,y) \geq k$ for all $x,y \in V(G)$. The edge connectivity of G equals the maximum *k* such that $\lambda'(x,y) \geq k$ for all $x,y \in V(G)$.