Graph Theory: Trees

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Trees and Spanning Trees

• A graph having no cycles is *acyclic*.

• A *forest* is an acyclic graph.

• A *leaf* is a vertex of degree 1.

• A *spanning sub-graph* of G is a sub-graph with vertex set V(G).

• A *spanning tree* is a spanning sub-graph that is a tree.
Distances

- If G has a $u,v$-path, then the distance from $u$ to $v$, written $d_G(u,v)$ or simply $d(u,v)$, is the least length of a $u,v$-path.
  - If G has no such path, then $d(u,v) = \infty$
Tree: Characterization

• An n-vertex graph G (with n \( \geq 1 \)) is a tree iff:
  
  – G is connected and has no cycles
  
  – G is connected and has n–1 edges
  
  – G has n–1 edges and no cycles
  
  – For \( u, v \in V(G) \), G has exactly one \( u, v \)-path
Some results …

• Every tree with at least two vertices has at least two leaves.
  – Deleting a leaf from a tree with $n$ vertices produces a tree with $n-1$ vertices.

• If $T$ is a tree with $k$ edges and $G$ is a simple graph with $\delta(G) \geq k$, then $T$ is a sub-graph of $G$. 
Some results ...

- If $T$ and $T'$ are two spanning trees of a connected graph $G$ and $e \in E(T) - E(T')$, then there is an edge $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of $G$. 
Diameter and Radius

- The **eccentricity** of a vertex $u$, written $\varepsilon(u)$, is the maximum of its distances to other vertices.

- In a graph $G$, the **diameter**, $\text{diam}_G$, and the **radius**, $\text{rad}_G$, are the maximum and minimum of the vertex eccentricities respectively.

- The **center** of $G$ is the subgraph induced by the vertices of minimum eccentricity.
Counting Trees

- There are $n^{n-2}$ trees with vertex set $[n]$. 
Prüfer Code / Sequence

Algorithm: \( \text{Production of } f(T) = \{a_1, \ldots, a_{n-2}\} \)

Input: A tree \( T \) with vertex set \( S \subseteq \mathbb{N} \).

Iteration: At the \( i^{th} \) step, delete the least remaining leaf, and let \( a_i \) be the neighbor of this leaf.