Graph Theory: Some Counting Problems

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Degree

- The *degree* of a vertex *v* in a graph G, written as $d_G(v)$ or simply $d(v)$, is the number of non-loop edges containing *v* plus twice the number of loops containing *v*.
  - The maximum degree is $\Delta(G)$
  - The minimum degree is $\delta(G)$
  - A graph is *regular* if $\Delta(G) = \delta(G)$
  - A graph is *k-regular* if $\Delta(G) = \delta(G) = k$
  - A vertex of degree *k* is *k-valent*.

Order and Size

- The *order* of a graph G, written as *n(G)*, is the number of vertices in G.

- The *size* of a graph G, written as *e(G)*, is the number of edges in G.

Countings and Bijections

- If G is a graph, then $\sum_{v \in V(G)} d(v) = 2e(G)$

- Every graph has an even number of vertices of odd degree.
  - No graph of odd order is regular with odd degree

- A k-regular graph with *n* vertices has *nk/2* edges.

Countings and Bijections

- Suppose *J,H,G* are graphs with J $\subseteq$ H $\subseteq$ G, and suppose H contains *q* copies of J. If G contains *m* copies of J, and the i^{th} copy of J appears in k_i copies of H in G, then G contains $\Sigma_{i=1,m}\ k_i / q$ copies of H.

- For n≥1, there are $2^{\binom{n-1}{2}}$ simple graphs with vertex set $\{v_1, \ldots, v_n\}$ such that every vertex degree is even.

Pigeon-hole Principle

If a set consisting of more than *kn* objects is partitioned into *n* classes, then some class receives more than *k* objects.

- Every simple graph with at least two vertices has two vertices of equal degree.

- If G is a simple *n*-vertex graph with $\delta(G) \geq (n-1)/2$, then G is connected

Turan Graph

- The *Turan graph* $T_{n,r}$ is the complete r-partite graph with *n* vertices that has *b* parts of size *a+1* and *r–b* parts of size *a*, where $a = n/r$ and $b = n - ra$.

- Turan showed that $T_{n,r}$ is the unique largest simple *n*-vertex graph with no r+1-clique.

Mantel's Theorem & Turan's Theorem

- [Mantel 1907] The maximum number of edges in an *n*-vertex triangle-free graph is: $\lfloor n^2/4 \rfloor$

- [Turan 1941] Among the *n*-vertex simple graphs with no r+1-clique, $T_{n,r}$ has the maximum number of edges.