Resolution Refutation Proofs

Course: CS40002
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The basic steps

- Convert the set of rules and facts into clause form (conjunction of clauses)
- Insert the negation of the goal as another clause
- Use resolution to deduce a refutation

If a refutation is obtained, then the goal can be deduced from the set of facts and rules.

Conversion to Normal Form

- A formula is said to be in clause form if it is of the form:

$$\forall x_1 \ \forall x_2 \ \ldots \ \forall x_n \ [C_1 \wedge C_2 \wedge \ldots \wedge C_k]$$

- All first-order logic formulas can be converted to clause form
- We shall demonstrate the conversion on the formula:

$$\forall x \ \{p(x) \Rightarrow \exists z \ \{ \neg \forall y \ [q(x,y) \Rightarrow p(f(x_1))] \\ \wedge \forall y \ [q(x,y) \Rightarrow p(x)] \ \}\}$$

Conversion to Normal Form

- Step1: *Take the existential closure and eliminate redundant quantifiers.* This introduces $\exists x_1$ and eliminates $\exists z$, so:

$$\forall x \{p(x) \Rightarrow \exists z \{ \neg \forall y [q(x,y) \Rightarrow p(f(x_1))] \wedge \forall y [q(x,y) \Rightarrow p(x)] \}\}$$

$$\exists x_1 \forall x \{p(x) \Rightarrow \{ \neg \forall y [q(x,y) \Rightarrow p(f(x_1))] \wedge \forall y [q(x,y) \Rightarrow p(x)] \}\}$$

Conversion to Normal Form

- Step 2: *Rename any variable that is quantified more than once.* y has been quantified twice, so:

$$\exists x_1 \; \forall x \; \{p(x) \Rightarrow \{ \neg \forall y \; [q(x,y) \Rightarrow p(f(x_1))] \wedge \forall y \; [q(x,y) \Rightarrow p(x)] \}\}$$

$$\exists x_1 \; \forall x \; \{p(x) \Rightarrow \{ \neg \forall y \; [q(x,y) \Rightarrow p(f(x_1))] \wedge \forall z \; [q(x,z) \Rightarrow p(x)] \}\}$$

Conversion to Normal Form

- Step 3: *Eliminate implication.*

$$\exists x_1 \; \forall x \; \{p(x) \Rightarrow \{ \neg \forall y \; [q(x,y) \Rightarrow p(f(x_1))] \wedge \forall z \; [q(x,z) \Rightarrow p(x)] \}\}$$

$$\exists x_1 \; \forall x \; \{\neg p(x) \vee \{ \neg \forall y \; [\neg q(x,y) \vee p(f(x_1))] \wedge \forall z \; [\neg q(x,z) \vee p(x)] \}\}$$

Conversion to Normal Form

- Step 4: *Move ¬ all the way inwards.*

$$\exists x_1\ \forall x\ \{\neg p(x) \vee \{\ \neg\forall y\ [\neg q(x,y) \vee p(f(x_1))]\ \wedge \forall z\ [\neg q(x,z) \vee p(x)]\ \}\}$$

$$\exists x_1\ \forall x\ \{\neg p(x) \vee \{\exists y\ [q(x,y) \wedge \neg p(f(x_1))]\ \wedge \forall z\ [\neg q(x,z) \vee p(x)]\ \}\}$$

Conversion to Normal Form

- Step 5: *Push the quantifiers to the right.*

$$\exists x_1 \; \forall x \; \{\neg p(x) \vee \{\exists y \; [q(x,y) \wedge \neg p(f(x_1))] \wedge \forall z \; [\neg q(x,z) \vee p(x)] \}\}$$

$$\exists x_1 \; \forall x \; \{\neg p(x) \vee \{[\exists y \; q(x,y) \wedge \neg p(f(x_1))] \wedge [\forall z \; \neg q(x,z) \vee p(x)] \}\}$$

Conversion to Normal Form

- Step 6: *Eliminate existential quantifiers (Skolemization).*
  - Pick out the leftmost $\exists y\, B(y)$ and replace it by $B(f(x_{i1}, x_{i2},\ldots, x_{in}))$, where:
    a) $x_{i1}, x_{i2},\ldots, x_{in}$ are all the distinct free variables of $\exists y\, B(y)$ that are universally quantified to the left of $\exists y\, B(y)$, and
    b) F is any n-ary function constant which does not occur already

Conversion to Normal Form

- Skolemization:

$$\exists x_1 \; \forall x \; \{\neg p(x) \lor \{[\exists y \; q(x,y) \land \neg p(f(x_1))] \land [\forall z \; \neg q(x,z) \lor p(x)] \}\}$$

$$\forall x \; \{\neg p(x) \lor \{[q(x,g(x)) \land \neg p(f(a))] \land [\forall z \; \neg q(x,z) \lor p(x)] \}\}$$

Conversion to Normal Form

- Step 7: *Move all universal quantifiers to the left*

$$\forall x \, \{\neg p(x) \vee \{[q(x,g(x)) \wedge \neg p(f(a))] \wedge [\forall z \, \neg q(x,z) \vee p(x)]\}\}$$

$$\forall x \, \forall z \, \{\neg p(x) \vee \{[q(x,g(x)) \wedge \neg p(f(a))] \wedge [\neg q(x,z) \vee p(x)]\}\}$$

Conversion to Normal Form

- Step 8: *Distribute* $\wedge$ over $\vee$.

$$\forall x\ \forall z\ \{[\neg p(x) \vee q(x,g(x))] \wedge [\neg p(x) \vee \neg p(f(a))] \wedge [\neg p(x) \vee \neg q(x,z) \vee p(x)]\ \}$$

- Step 9: (Optional) *Simplify*

$$\forall x\ \{[\neg p(x) \vee q(x,g(x))]\ \wedge\ \neg p(f(a))\ \}$$

Resolution

- If Unify(z_j, $\neg q_k$) = θ, then:

$$\frac{z_1 \vee \ldots \vee z_m, \qquad q_1 \vee \ldots \vee q_n}{\mathrm{SUBST}(\theta, z_1 \vee \ldots \vee z_{i-1} \vee z_{i+1} \vee \ldots \vee z_m \\ \vee q_1 \vee \ldots \vee q_{k-1} \vee q_{k+1} \vee \ldots \vee q_n)}$$

Example

- Harry, Ron and Draco are students of the Hogwarts school for wizards
- Every student is either wicked or is a good Quidditch player, or both
- No Quidditch player likes rain and all wicked students like potions
- Draco dislikes whatever Harry likes and likes whatever Harry dislikes
- Draco likes rain and potions
- Is there a student who is good in Quidditch but not in potions?

Resolution Strategies

- Unit Resolution
  - Every resolution step must involve a unit clause
  - Leads to a good speedup
  - Incomplete in general
  - Complete for Horn knowledge bases

Resolution Strategies

- Input Resolution
  - Every resolution step must involve a a input sentence (from the query or the KB)
  - In Horn knowledge bases, Modus Ponens is a kind of input resolution strategy
  - Incomplete in general
  - Complete for Horn knowledge bases

Resolution Strategies

- Linear Resolution
  - Slight generalization of input resolution
  - Allows P and Q to be resolved together either if P is in the original KB, or if P is an ancestor of Q in the proof tree
  - Linear resolution is complete