INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

Date: .... *FN/AN* Time: *3 Hrs* Full Marks: *100* No. of students: 10
Autumn End Semester Exams, 2004 Dept: *Computer Sc. & Engg.* Sub No: *CS60047*
M.Tech (Elective) Sub Name: *Advanced Graph Theory*

Instructions: *Answer Question 1 and any four from the rest*
All parts of a question must be answered in the same place

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1. (a) Consider the graphs $G_1 = \overline{C_6}$, $G_2 = \overline{C_3 + C_3}$, $G_3 = K_3 \vee K_3$ and $G_4 = Q_3$.
   i. Which of these graphs are planar? Give a planar drawing of the planar ones and a justification for the nonplanar ones.
   ii. Write down the chromatic numbers of these graphs.
   iii. Write down the vertex connectivities of these graphs.

   [8+4+4= 16 marks]

   (b) What is the maximum number of strongly connected components having more than one vertex in a simple acyclic digraph of n vertices and e edges? [2 marks]

   (c) Let G be a connected simple graph having n vertices and e edges that does not have P_4 or C_4 as an induced subgraph. Determine $\Delta(G)$ with justification. [4 marks]

   (d) Demonstrate that two planar embeddings of a planar graph may have nonisomorphic duals.

   [3 marks]

   (e) Indicate which of the following statements are correct. Give very short (informal) arguments for the correct ones and small counterexamples for the incorrect ones.
   i. A graph with every vertex degree even has no cut-edge.
   ii. The line graph of a planar graph is also planar.
   iii. Q_k (the k-dimensional hypercube) is Hamiltonian.
   iv. If a tournament is strongly connected, then it has more than one king.
   v. If $e(G) \leq 3n(G) - 6$, then G is 4-colorable.

   [5 $\times$ 3 = 15 marks]

2. (a) Prove that $\chi(G) \times \chi(\overline{G}) \geq n(G)$.

   (b) Consider the Greedy coloring algorithm with the vertices selected in non-increasing order of their degrees. Draw a graph for which this algorithm will *not* yield an optimal coloring.

   [8+7 = 15 marks]

3. G is a simple non-trivial graph.

   (a) Prove or disprove: $\alpha(G) \leq n(G) - e(G)/\Delta(G)$

   (b) Prove that $\alpha(G) \leq n(G)/2$ when G is regular

   [8+7 = 15 marks]

4. A country has a set V of cities. The highway map of the country is a graph $G = (V, E)$, where the edges in E represent highways between the cities. The air map of the country is another graph $H = (V, E')$, where an edge between two vertices indicates that there exists one or more direct flights between the cities (in both directions). A city x is said to be *air-connected* to a city y if one can travel from y to x by air (possibly by changing flights in intermediate cities). Assume that the cost of air travel between every pair of air-connected cities is constant (even if it involves one or more hops).

   A highway inspection team intends to inspect every highway by traversing once along each highway. They can also travel from one city to another by air, provided that these cities are air-connected.

   (a) Give an algorithm to determine whether there exists a tour for the highway inspection team (including intermediate flights) that covers all the highways exactly once.

   (b) Extend your algorithm to find a tour (if it exists) that covers all highways exactly once and minimizes the total air-fare.

   Your algorithm may contain calls to standard algorithms. You must prove the correctness of your algorithm.

   [8+7 = 15 marks]

5. (a) Prove that the compliment of planar graphs having more than 6 vertices cannot be planar. Use only Euler's formula as a known result.

   (b) Let G be a connected 3-regular plane graph in which every vertex lies on one face of length 4, one face of length 6, and one face of length 8. In terms of $n(G)$, determine the number of faces of each length.

   [8+7 = 15 marks]

6. (a) State the Max-flow Min-cut theorem (due to Ford-Fulkerson). Give an example where an arbitrary choice of augmenting paths may lead to an exponential number of augmentations.

   (b) Several countries send athletes to a competition; the i^{th} country sends m_i athletes. The organizers of the competition conduct several simultaneous heats (prelims) in groups; the j^{th} group can accomodate up to n_j participants. The organizers want to schedule all the participants into groups, but the participants from the same country must be in different groups. The groups need not all be filled. Formulate the problem as a network flow problem. You should specify the exact structure of your graph, the meaning of your vertices and edges, and the edge capacities.

   [8+7 = 15 marks]

7. Write short notes on the following:

   (a) Kuratowski's Theorem: *Statement and proof outline*.

   (b) Matchings in unweighted non-bipartite graphs.

   (c) Hamiltonian Closure: *Definition and proof of uniqueness*.

   [5+5+5 = 15 marks]