Reducibility

CS60001: Foundations of Computing Science

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Undecidable Problems from Language Theory

- **Theorems:**
  - Let $HALT_{TM}$ = {<M, w> | M is a TM and M halts on input w},
    then $HALT_{TM}$ is *undecidable*
  - Let E_{TM} = {<M> | M is a TM and L(M) = Φ},
    then E_{TM} is *undecidable*
  - Let $REGULAR_{TM}$ = {<M> | M is a TM and L(M) is a regular language},
    then $REGULAR_{TM}$ is *undecidable*
  - Let EQ_{TM} = {<M_1, M_2> | M_1 & M_2 are TMs and $L(M_1) = L(M_2)$},
    then EQ_{TM} is *undecidable*

Reductions via Computation Histories

- **Important Definitions:**
  - Let M be a Turing machine and w an input string. An *accepting computation history* for M on w is a sequence of configurations, $C_1, C_2, \ldots, C_l$, where C_1 is the start configuration of M, and each C_i legally follows from C_{i-1} according to the rules of M. A *rejecting computation history* for M on w is defined similarly, except that C_l is a rejecting configuration.
  - A *linear bounded automaton (LBA)* is a restricted type of Turing machine wherein the tape head is not permitted to move off the portion of the tape containing the input. If the machine tries to move its head off either end of the input, the head stays where it is, in the same way that the head will not move off the left-hand end of an ordinary Turing machine's tape.

Reductions via Computation Histories (contd…)

- **Lemma:**
  - Let M be an LBA with q states and g symbols in the tape alphabet. There are exactly qng^n distinct configurations of M for a tape of length n

- **Theorems:**
  - Let A_{LBA} = {<M, w> | M is an LBA that accepts string w}, then A_{LBA} is *decidable*
  - Let E_{LBA} = {<M> | M is an LBA where L(M) = Φ}, then E_{LBA} is *undecidable*
  - Let ALL_{CFG} = {<G> | M is a CFG and L(G) = Σ*}, then ALL_{CFG} is *undecidable*

A Simple Undecidable Problem

- **Post Correspondence Problem (PCP)**
  - The PCP is a collection P of dominos:
    - $P = \{ [\frac{t_1}{b_1}], [\frac{t_2}{b_2}], \ldots, [\frac{t_k}{b_k}] \}$
  - A match is a sequence $i_1, i_2, \ldots, i_l$; where $t_{i1}\, t_{i2} \ldots t_{il} = b_{i1}\, b_{i2} \ldots b_{il}$
  - The problem is to determine whether P has a *match*

- **Theorem:**
  - Let PCP = {<P> | P is an instance of the Post correspondence problem with a match}, then PCP is *undecidable*

Mapping Reducibility

- A function $f: \Sigma^* \rightarrow \Sigma^*$ is a <u>*computable function*</u> if some Turing machine *M*, on every input *w*, halts with just *f(w)* on its tape
- Language *A* is mapping reducible to language *B*, written $A \leq_m B$,
  - If there is a computable function $f: \Sigma^* \rightarrow \Sigma^*$, where for every *w*, $w \in A \Rightarrow f(w) \in B$
  - The function *f* is called the *reduction of A to B*
- Theorems:
  - If $A \leq_m B$ and *B* is decidable, the *A* is *decidable*
  - If $A \leq_m B$ and *A* is undecidable, the *B* is *undecidable*
  - If $A \leq_m B$ and *B* is Turing-recognizable, the *A* is *Turing-recognizable*
  - If $A \leq_m B$ and *A* is not Turing-recognizable, the *B* is not *Turing-recognizable*
  - Let $EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ \& } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$, then EQ_{TM} is *neither Turing-recognizable, nor co-Turing-recognizable*