

Model Single Server Queues with Exponential Service Time and Poisson Arrivals

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Analysis of a Simple Queue

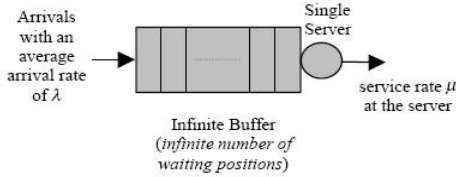


Fig. 1. Analysis of single server Queue

The queue we are analyzing is the single server M/M/1/∞ queue with Poisson arrivals, exponentially distributed service times and infinite number of buffer positions

I. Assumptions

- The arrival process is a Poisson process with exponentially distributed random inter-arrival times
- The service time is an exponentially distributed random variable
- The arrival process and the service process are independent of each other

II. Calculations

Arrival Process

$$\text{Mean Inter-arrival time} = \frac{1}{\lambda}$$

Service Process

$$\text{Mean Service time} = \frac{1}{\mu}$$

Assume that, as $\Delta t \rightarrow 0$

$$P\{\text{one arrival in time } \Delta t\} = \lambda \Delta t$$

$$P\{\text{no arrival in time } \Delta t\} = 1 - \lambda \Delta t$$

$$P\{\text{more than one arrival in time } \Delta t\} = O((\Delta t)^2) = 0$$

$$P\{\text{one departure in time } \Delta t\} = \mu \Delta t$$

$$P\{\text{no departure in time } \Delta t\} = 1 - \mu \Delta t$$

$$P\{\text{more than one departure in time } \Delta t\} = O((\Delta t)^2) = 0$$

$$P\{\text{one or more arrival and one or more departure in time } \Delta t\} = O((\Delta t)^2) = 0$$

The state of the queue is defined by defining an appropriate system state variable

System State at time $t = N(t) =$ Number in the system at t (waiting and in service)

Let $p_N(t) = P\{\text{system in state } N \text{ at time } t\}$

By ignoring terms with $(\Delta t)^2$ and higher order terms, the probability of the system state at time $t + \Delta t$ may then be found as -

$$p_0(t + \Delta t) = p_0(t)[1 - \lambda \Delta t] + p_1(t)\mu \Delta t \quad N = 0$$

$$p_N(t + \Delta t) = p_N(t)[1 - \lambda \Delta t - \mu \Delta t] + p_{N-1}(t)\lambda \Delta t + p_{N+1}(t)\mu \Delta t \quad N > 0$$

subject to the normalisation condition that

$$\sum_{\forall i} p_i(t) = 1 \text{ for all } t \geq 0$$

Taking the limits as $\Delta t \rightarrow 0$, and subject to the same normalization, we get

$$\begin{aligned} \frac{dp_0(t)}{dt} &= -\lambda p_0(t) + \mu p_1(t) \\ \frac{dp_N(t)}{dt} &= -(\lambda + \mu)p_N(t) + \lambda p_{N-1}(t) + \mu p_{N+1}(t) \end{aligned}$$

These equations may be solved with the proper initial conditions to get the Transient Solution. If the queue starts with N in the system, then the corresponding initial condition will be

$$\begin{aligned} p_i(0) &= 0 \text{ for } i \neq N \\ p_N(0) &= 1 \text{ for } i = N \end{aligned}$$

For the equilibrium solution, the conditions invoked are -

$$\frac{dp_i(t)}{dt} = 0$$

and

$$p_i(t) = p_i \text{ for } i=0, 1, 2, \dots, \infty$$

For this, defining $\rho = \lambda/\mu$ erlangs, with $\rho \leq 1$ for stability, we get

$$\begin{aligned} p_1 &= \rho p_0 \\ p_{N+1} &= (1 + \rho)p_N - \rho p_{N-1} = \rho p_N = \rho_{N+1} p_0 \\ N &\geq 1 \end{aligned}$$

Applying the Normalization Condition $\sum_{i=0}^{\infty} p_i = 1$ we get

$$p_i = \rho^i (1 - \rho) \quad i = 0, 1, \dots, \infty$$

as the equilibrium solution for the state distribution when the arrival and service rates are such that $\rho = \lambda/\mu < 1$ The equilibrium solution does not depend on the initial condition but requires that the average arrival rate must be less than the average service rate

III. Mean Performance Parameters of Queue

(a) Mean Number in System, N

$$N = \sum_{i=0}^{\infty} i p_i = \sum_{i=0}^{\infty} i \rho^i (1 - \rho) = \frac{\rho}{1 - \rho}$$

(b) Mean Number Waiting in Queue, N_q

$$\begin{aligned} N_q &= \sum_{i=1}^{\infty} (i - 1) p_i \\ &= \frac{\rho}{1 - \rho} - (1 - p_0) \\ &= \frac{\rho}{1 - \rho} - \rho \\ &= \frac{\rho^2}{1 - \rho} \end{aligned}$$

Mean Performance Parameters of the Queue (c) Mean Time Spent in System W This would require the following additional assumptions

- FCFS system though the mean results will hold for any queue where the server does not idle while there are customers in the system
- The equilibrium state probability p_k will also be the same as the probability distribution for the number in the system as seen by an arriving customer
- The mean residual service time for the customer currently in service when an arrival occurs will still be $1/\mu$ Memory-less Property satisfied only by the exponential distribution

Using these assumptions, we can write

$$W = \sum_{k=0}^{\infty} \frac{(k + 1)}{\mu} p_k = \frac{1}{\mu(1 - \rho)}$$

(d) Mean Time Spent Waiting in Queue W_q

This will obviously be one mean service time less than W

$$W_q = W - \frac{1}{\mu} = \frac{\rho}{\mu(1 - \rho)}$$

(e) P{Arriving customer has to wait for service} = $1 - p_0 = \rho$

(f) Server Utilization Fraction of time the server is busy = $P\{\text{server is not idle}\} = 1 - p_0 = \rho$

IV. Example

Customers arrive as per Poisson distribution with mean rate of arrival of 30/hr. Required time to serve a customer has an Exponential distribution and is 90 sec. Determine queue characteristics: N , N_q , W , W_q

$$\begin{aligned}
 \lambda &= 30 \text{customer/hr} \\
 &= 30 \text{customer}/60 \text{min} \\
 &= \frac{1}{2} \text{customer/min} \\
 \frac{1}{\mu} &= 90 \text{sec/customer} * (1 \text{min}/60 \text{sec}) \\
 &= \frac{3}{2} \text{min/customer} \\
 \rho &= \lambda/\mu \\
 &= \frac{(1/2)}{(2/3)} \\
 &= \frac{3}{4} \\
 N &= \frac{\rho}{1 - \rho} \\
 &= \frac{3/4}{1 - 3/4} \\
 &= 3 \\
 N_q &= \frac{\rho^2}{1 - \rho} \\
 &= \frac{(3/4)^2}{1 - 3/4} \\
 &= 2.25 \\
 W &= \frac{1}{\mu(1 - \rho)} \\
 &= \frac{1}{2/3 * (1 - 3/4)} \\
 &= 6 \\
 W_q &= \frac{\rho}{\mu(1 - \rho)} \\
 &= \frac{3/4}{2/3 * (1 - 3/4)} \\
 &= 4.5
 \end{aligned}$$