Edge & Vertex Connectivity and Network Flow Problem

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I. Introduction

In this class, we will first continue the discussion of edge and vertex connectivity and learn some of the important results on them. Then we shall state the problem of Network Flow and present the famous Max-Flow Min-Cut Theorem.

II. Edge & Vertex Connectivity

In the last class we discussed the concept of connectivity in graphs. Here we will study some of the basic theorems regarding them. But let us first discuss the motivation behind studying connectivity. Suppose, we are given a network of computing nodes or machines and the aim of the network is to perform high performance computing. If their are some node or link failures, the network might get disconnected, thereby rendering parallel computing infeasible. Thus, it would thus be desirable that the network has a very high edge connectivity (to tolerate link failures) and vertex connectivity (to tolerate node failures). Thus, in general, we try to maximize the connectivities in a given network. This can be stated as an optimization problem.

Best Configuration Design Problem: Given n vertices and e edges, find the graph with maximum vertex connectivity and edge connectivity.

Theorem 2.1: The edge connectivity of a graph G cannot exceed the minimum degree of any node in G.

Proof: Consider the node v with the minimum degree in the graph G. Let the degree of v be k. If we delete all those k edges whose one end point is v, we get a disconnected graph. Thus,

$$k'(G) \leq k.$$

Theorem 2.2: The vertex connectivity of a graph G can never exceed the edge connectivity of G.

Proof: This can be proved as follows. Given any edge cut [S, S'] of size k, we can find a vertex cut set of size k. It can be obtained by removing all the vertices that are the end points of one of the edges in [S, S'] keeping in mind not to make S or S' null. Thus number of vertices removed is at most the number of edges in the edge cut. Thus,

$$k(G) \leq k'(G).$$

Also, since in any graph the average degree of a node is $2e/n$ thus minimum degree say $k \leq \lfloor\ 2e/n\ \rfloor$. Thus,

$$k(G) \leq k'(G) \leq k \leq \lfloor 2e/n \rfloor$$

Here, n: no. of vertices, e: no. of edges, $\lfloor$ x $\rfloor$: the greatest integer smaller than x.

Definitions: A connected graph G is *k-connected* iff every pair of vertices in G is joined by atleast k non-intersecting paths and there exists atleast one pair with exactly k non-intersecting paths. That is, $k(G) = k$. A connected graph G is *k-edge connected* iff every pair of vertices in G is joined by atleast k pairwise edge-disjoint paths and there exists atleast one pair with exactly k pairwise edge-disjoint paths. That is, $k'(G) = k$.

III. Network Flow

Definition: A *network* is a digraph with a non-negative capacity $c(e)$ on each edge e and a distinguished source vertex s and sink vertex t. A flow f assigns a value $f(e)$ to each edge e. We write $f^+(v)$ for the total flow on edges leavng v and $f^-(v)$ for the total flow on edges entering v. A transport network is shown in Figure 1. A flow is feasible if it satisfies the *capacity constraints* i.e. for each e,

$$0 \leq f(e) \leq c(e)$$

and the *conservation contraints* i.e. for each node v except for s and t,

$$f^+(v) = f^-(v)$$

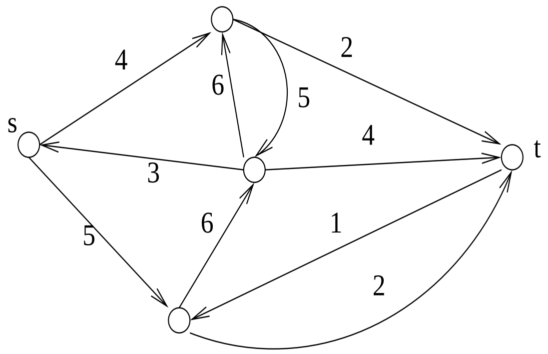

Fig. 1. A Transport Network. The numbers on the edges are the edge capacities and arrow indicates the direction of the flow

The efficiency of a given network is given by -

$$Efficiency = fmax/C$$

Here, C is the total capacity of the network.

Theorem 2.3 (Max-flow Min-cut Theorem): In a given transport network G, the maximum value of a feasible flow f from S to T is equal to the minimum value of any cut in G that separates S and T.

This can be proved using the properties of cut-sets and the capacity and conservation constraints on the flow. Please refer to any standard textbook for a complete proof.