Planar Graphs, Dual Graphs, Connectivity and Separability

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I. Introduction

In this class, we shall study some of the basic concepts of graph theory - Planar graphs & their properties, and vertex & edge connectivity in a graph.

II. Planar Graphs

Definitions: A graph is *planar* if it has a drawing without crossings. Such a drawing is a *planar embedding* of G. A *plane* graph is a particular planar embedding of a planar graph.

Planar graphs find applications in many fields. For example-

- In VLSI layouts, it is required that no two wires cross each, thereby creating a planar graph.
- Planar graphs are also useful for graph visualization purposes.

A graph that is not a planar graph is known as a *non-planar graph.* Let us next discuss some of the properties of non-planar graphs. The smallest possible non-planar graphs are -

- In terms of number of vertices - K_5
- In terms of number of edges - $K_{3,3}$

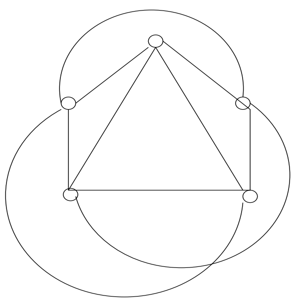

Fig. 1. K_5

Also, note that any planar graph can be represented by straight lines only.

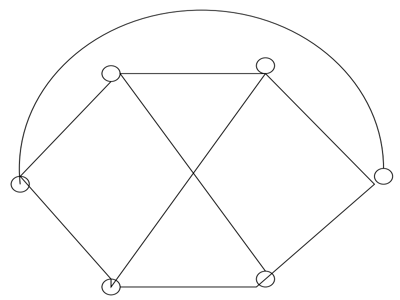

Fig. 2. $K_{3,3}$

A. Euler's Formula

Theorem 1.1: If a connected plane graph G has exactly n vertices, e edges, and f faces, then n - e + f = 2

Proof: We use induction on n.

Basis step (n=1): G only contain loops. Each loop encompasses a face. But there is an external face too. Thus, $f = e + 1$. Also, $n = 1$. Thus, n - e + $f = 2$.

Induction step ($n > 1$): Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G' with n' vetices, e' edges, and f' faces. The contraction, however, doesn't change the number of faces (we merely shortened boundaries), but it reduces the number of edges and vertices by 1, so $n' = n$-1, $e' = e$-1, and $f' = f$. Applying the induction hypothesis yields -

$$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2.$$

Theorem 1.2: In a simple connected planar graph, $e \geq 3f/2$ and $e \leq 3n$ -6

Proof: In a simple planar graph, any face boundary contains atleast 3 edges. Also, total face length is $2e$. Thus, we get that $2e \geq 3f$ (since every face length is atleast 3). Substituing this in the equation n - e + f = 2, we get,

$$n - e + 2e/3 \geq 2$$

or,

$$e \leq 3n - 6$$

B. Detection of Non-Planar Graphs

1. Using the theorem 1.2, if $e > 3n$ - 6 then the graph is non-planar.
2. Simply the graph using the following three steps-
   - Step 1: Remove all the parallel edges and loops.
   - Step 2: Any vertex with degree 2 is replaced by a single edge connecting the other two vertices.

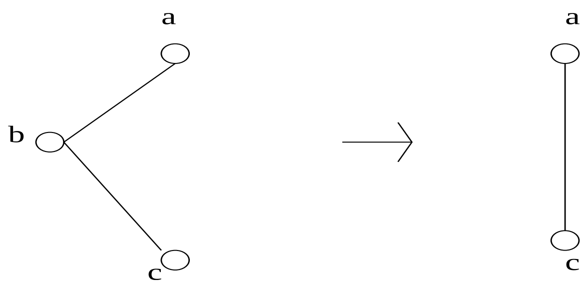

Fig. 3. Step 2

- Step 3: Repeat Steps 1 and 2 until no more changes occur.

If the resultant graph contains K_5 or $K_{3,3}$ then it is non-planar.

C. Dual Graphs

Definition: The *dual graph* G* of a plane graph G is a plane graph whose vertices correspond to the faces of G. The edges of G* correspond to the edges of G as follows: if e is an of G with face X on one side and face Y on the other side, then the endpoints of the dual edge e* are the vertices x, y that represent the faces X, Y of G. This concept is only valid for plane graphs.

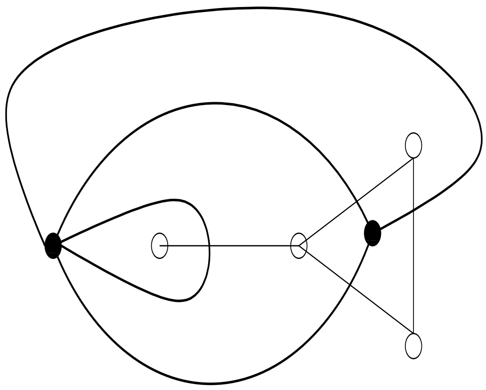

Fig. 4. The black nodes and the edges among them belongs to the dual graph

III. Cut-Sets and Connectivity

Definitions: A *separating set* or *vertex cut* of a graph G is a set S, a subset of $V(G)$ such that G - S has more than one component. The *connectivity* of G, written k(G), is the minimum size of vertex set S such that G - S is disconnected or has only one vertex.

Edge Cut-Sets and Edge Connctivity

Definitions: A *disconnecting set* of edges is a set F, a subset of E(G) such that G - F has more than one component. The *edge connectivity* of G, written k'(G), is the minimum size of a disconnecting set.

Definition: A graph is called a *separable graph* if its vertex-connectivity is equal to 1. That node is called the *articulation-point* or the *cut-point*.