Assignment 6a - 21.04.2007

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I. Problem

Given n nodes with an average degree d, you can make a graph where each node had exactly d edges and where each node n_i is connected to its d closest neighbors. Such a graph is called a $regular graph$.

1) If n=5 and d=3 it is impossible to construct such a graph. Why?
2) Explain the general case where a regular graph cannot be constructed based on the properties / relationship of n and d.
3) Prove the statement that you make in (2)

1) Answer 1.1

Since the average degree d of the graph is 3 and the number of nodes n is 5 and the number of edges in the graph e is given by

$$e = \frac{n \cdot d}{2} \tag{1}$$

Hence, the number of edges for the graph given by the formula comes out to be $(15/2)$ and by common knowledged we know that the number of edges in a graph cannot be fractional.

2) Answer 1.2

Generalizing the above justification we can claim that there exists no regular graph taht has odd number of edges and odd number of vertices.

3) Answer 1.3

A simple proof by contradiction for the generalization in *subsection I-.2* is given using the *Equation 1* with a justification same as the one given in *subsection I-.1*.

Let us assume there exists a regular graph with odd number of edges and odd degree. Let the number of nodes the regular be n be written as $\exists i, n = 2i{+}1$ and the degree of each node d be written as $\exists j, d = 2j{+}1$, where i and j can be either can be a non-negative integer (odd or even). Then, the *Equation 1* changes to

$$\exists i \exists j, \quad i,j \in Z^+ \quad e = \frac{(2i+1)\cdot(2j+1)}{2} \tag{2}$$

$$\exists i \exists j, \quad i,j \in Z^+ \quad e = \frac{4ij + 2(i+j) + 1}{2} \tag{3}$$

$$\exists i \exists j, \quad i,j \in Z^+ \quad e = 2ij + (i+j) + \frac{1}{2} \tag{4}$$

$$\exists k, \quad k \in Z^+, \quad k = 2ij + (i+j) \quad e = k + \frac{1}{2} \tag{5}$$

From *Eqn. 5*, the number of edges in a regular graph with odd number of nodes and odd degree comes out to be a integer + $\frac{1}{2}$ which is not possible. Hence, by contradiction, there cannot exist such a graph.

II. Problem

In the graph given in *Fig. II*, there are 16 nodes. The graph is undirected and regularly connected where the degree of each node $d = 3$.

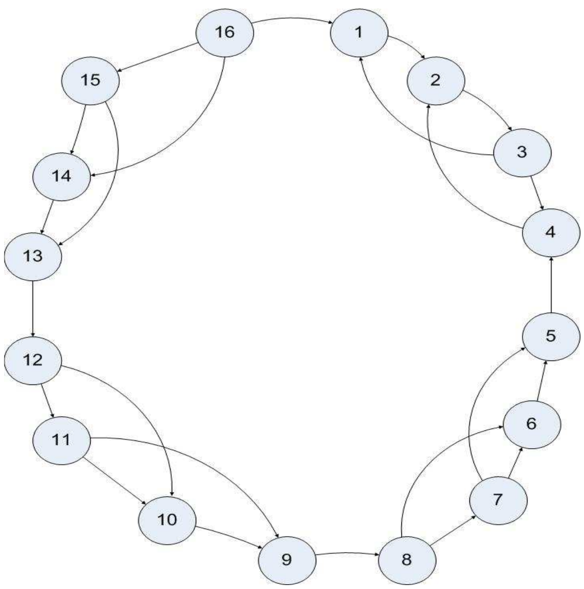


Fig. 1. Graph for *Problem II*

I have chosen to randomly rewire these edges as shown in the *Table II*.Each row represents a rewire, where the original edge is removed and replaced with the new edge. For example, if the original edge is (a, b) and the new edge is (g, j) that means I removed the existing edge between a and b and added in an edge from g to j.

1) What is the average degree after the graph has been rewired?
2) How did I "randomly" determine which edges to add in?
3) If I were to rewire one more pair, what edge follows in the pattern and why is that a problem?

Old Edge	New Edge
(1,3)	(3,14)
(5,7)	(15,9)
(16,1)	(2,6)
(2,3)	(5,3)
(4,5)	(5,8)
(6,7)	(9,7)
(8,9)	(9,3)
(10,11)	(2,3)
(12,13)	(8,4)

TABLE I

TABLE FOR *Problem II*

4) Answer 2.1

The new rewired graph looks like the *Fig. II-.4* and since the number of edges remains the same, the average degree of the graph doesn't get altered.

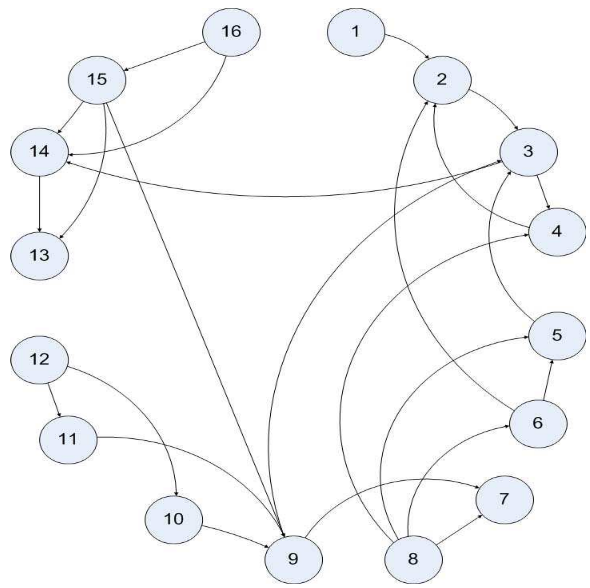


Fig. 2. Rewired Graph for *Problem II*

5) Answer 2.2

The nodes between which the edges were to be added were chosen randomly by giving a high probability to the nodes that already had a high degree (though initially everybody had the same degree, so had an equal chance of getting the new edge). This is why we see that the in the rewired graph, certain nodes like 9, 3 having higher degree than others.

6) Answer 2.3

The next edge to be added would be possibly between 3 and 8 (and removed between 13 and 14 as is the trend in removal). This system of *preferential* attachment may create singleton vertices and we will loose the connectivity in the graph eventually.

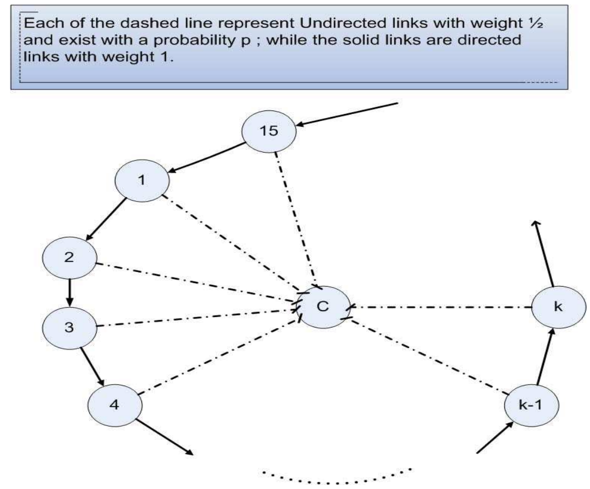
III. Problem

.Consider the following small world model. There are n nodes arranged on a directed ring. Every node is connected to the one on its left with a directed link of unit length. We add a new central point to the network, and connect it with every other vertex on the circle with probability p, using an undirected link of length $\frac{1}{2}$. Essentially, this means that we select some vertices with probability p and then connect them with an undirected link of length 1.

1) Compute the probability $P(l,k)$ that the shortest path from one vertex to another is l when the distance between the vertices along the ring is k.
   (Hint: $\sum_{l=1}^{k} P(l,k) = 1$)
2) Find the distribution $P(l)$ of the shortest path between pairs of vertices. Compute the average shortest path between the vertices of the graph.

7) Answer 3.1

Consider the *Figure III*. For a path of k along the ring there will be $k+1$ nodes. The shortest path will always take only one *short-cut* (undirected link created in the problem), since if it took two, then there already existed a *short-cut* between the first node of the first node of the first *short-cut* and the last node of the last *short-cut* which the path didn't take in the first place and hence the path would no longer be a shortcut. Hence, the problem of path of having a shortest path of length l is same as finding a path of length $l-k+1$ amongst the k linear nodes along the ring that is *shorted* by an undirected link



The probability for this to happen is choosing two nodes amongst $k+1$ linear nodes which are separated by $k-l$ nodes in between. For this, we will need to identify a node in the first l nodes of the $k+1$ nodes in going from 1 to $k+1$ (see *Fig. III-.7*). This is done with a probability of $\frac{l}{(k+1)}$. Then, for the two nodes where the shortcut begins, and ends their probability of having the shortcut is p each while the rest $l-1$ nodes will have to have a probability $(1-p)$. Hence, the probability is-

$$P(l,k) = \frac{l}{k+1} \cdot p^2 \cdot (1-p)^{l-1} \tag{6}$$

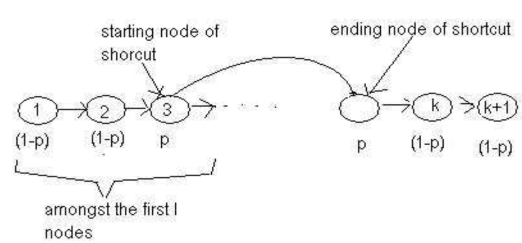


Fig. 3. Linear *k+1* nodes in the ring for *Problem III*

8) **Answer 3.2**

Once $P(l,k)$ is known, the distribution of $P(l)$ between pairs of vertices in given by substituting in $P(l,k)$, k from 1 through $\lceil \frac{n-1}{2} \rceil$. The average shortest path between the vertices is give by

$$Avg.ShotestPath = \sum_k = 1^{n-1} \sum_l = 1^k P(l,k) \quad (7)$$