Assignment V

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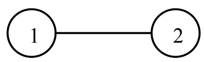
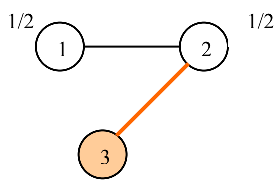
03CS3025

Prob. 1. Barabasi–Albert Model : probability that a new node will join to a k-degree existing node is proportional to k. (I have used a random number generator to select a node to which the new node will join according to its preference order.)

Step 0 :

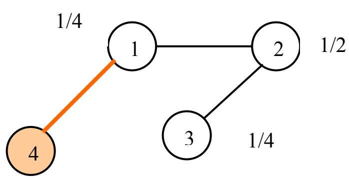


Step 1:



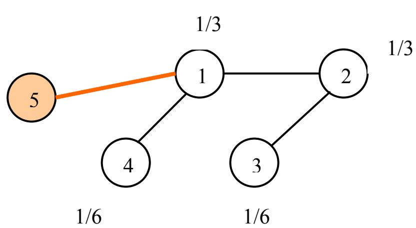
Selected through Random Number Generator (www.random.org)

Step 2:

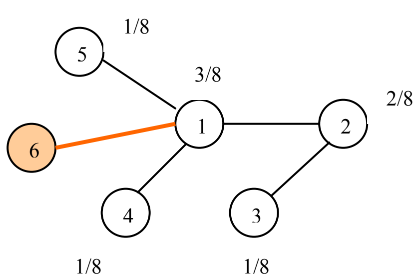


Selected through Random Number Generator (www.random.org)

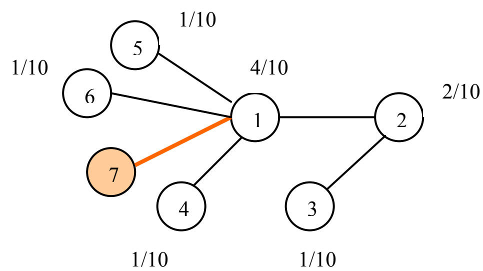
Step 3:



Step 4:



Step 5:



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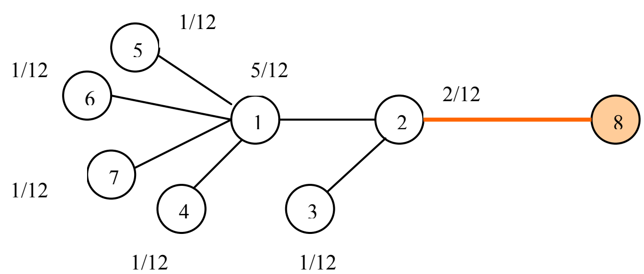
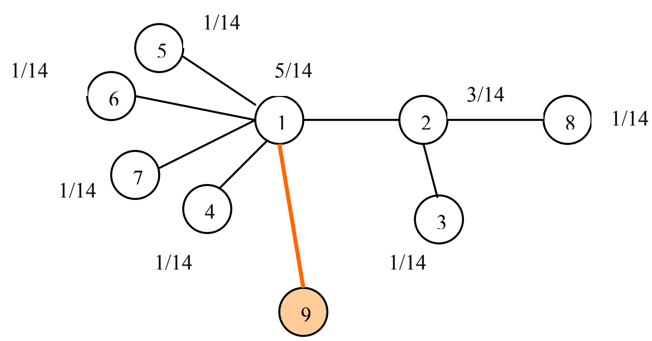
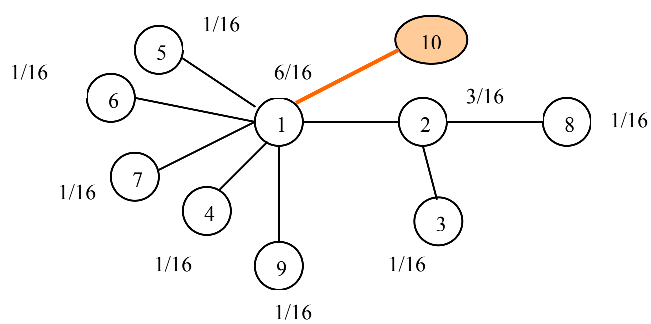


Step 7:



Step 8:



Final Degree Distribution is :

7	3	1	1	1	1	1	1	1	1

(as per node numbering)

Prob.2.

We have to prove this,

$$n_k \sim \begin{cases} k^{-\gamma}\exp\left[-\mu\left(\frac{k^{1-\gamma}-2^{1-\gamma}}{1-\gamma}\right)\right], & \frac{1}{2}<\gamma<1, \\ k^{(\mu^2-1/2)}\exp[-2\mu\sqrt{k}], & \gamma=\frac{1}{2}, \\ k^{-\gamma}\exp\left[-\mu\frac{k^{1-\gamma}}{1-\gamma}+\frac{\mu^2}{2}\frac{k^{1-2\gamma}}{1-2\gamma}\right], & \frac{1}{3}<\gamma<\frac{1}{2}, \end{cases}$$

We know that,

$$n_k = \frac{\mu}{A_k}\prod_{j=1}^{k}\left(1+\frac{\mu}{A_j}\right)^{-1}.$$

For Sub-Linear kernels which are asymptotically homogenous, $A_k \sim k^\gamma$

Now,

$$n_k = (\mu/ k^\gamma)\ \mathbf{\Pi}\ (1 + (\mu/ j^\gamma))^{-1}$$

writing the product in the form of exponential of a sum,

$$n_k = (\mu/ k^\gamma)\ \mathbf{\Pi}\ \mathbf{exp}(\mathit{log}(1 + (\mu/ j^\gamma))^{-1})$$

$$= (\mu/ k^\gamma)\ \exp(\Sigma\ \mathit{log}(1 + (\mu/ j^\gamma))^{-1})$$

$$= (\mu/ k^\gamma)\ \exp(-\Sigma\ \mathit{log}(1 + (\mu/ j^\gamma)))$$

$$= (\mu/ k^\gamma)\ \exp(- I)$$

Where, $I = \sum_{j=1}^{k} \mathit{log}(1 + (\mu/ j^\gamma))$

Not able to solve I !!!

Prob 3.

For a power-law for integer variables the following holds:

$$p(k) = C\,\mathrm{B}(k, \alpha) = C\,(\Gamma(k)\,\Gamma(\alpha)/\,\Gamma\,(k+\alpha))$$

where C is a normalizing constant and $\Gamma(x)$ is the standard Γ-function with the property $\Gamma(x+1) = x\,\Gamma(x)$.

Now,

$$I(a, b) = \int u^{a-1}(1-u)^{b-1}du$$

Using Integration by parts,

$$I(a, b) = [u^{a-1}(1-u)^{b-1}]/\mathrm{b}\,\Big|_0^1 + (1/\mathrm{b})\int_0^1 (\mathrm{a}-1)\; u^{a-2}(1-u)^{b}\, du$$

$$= 0 + (\mathrm{a}-1)/\mathrm{b}\ \mathrm{I}(\mathrm{a}-1,\mathrm{b}+1)$$

$$= ({}^{a-1}/_{b})I(a-1,b+1)$$

By repeated use of this formula,

$$\mathrm{I}(\mathrm{k}, \alpha) = (\mathrm{k}-1/\,\alpha\,)\, I(\mathrm{k}-1, \alpha+1)$$

$$= (\mathrm{k}-1/\,\alpha)(k-2/\,\alpha+1)\, I(\mathrm{k}-2, \alpha+2)$$

$$= (\mathrm{k}-1/\,\alpha)(k-2/\,\alpha+1)(k-3/\,\alpha+2)\ldots\ldots$$

Multiplying denominator and numerator by $(\alpha-1)(\alpha-2)(\alpha-3)\ldots..1$, we get

$$= (\Gamma(k)\,\Gamma(\alpha)/\,\Gamma\,(k+\alpha))$$

$$= \mathrm{B}(k, \alpha)$$

Hence,

$$\mathrm{B}(k, \alpha) = \int_0^1 u^{k-1}(1-u)^{\alpha-1}du$$

Now,

$\sum p(k) = 1$

Hence,

$\sum C\,\mathrm{B}(k, \alpha) = 1$

i.e; $C \sum \mathrm{B}(k, \alpha) = 1$

i.e; $C\,[\sum \int u^{k-1}(1-u)^{\alpha-1} du\,] = 1$

i.e; $C\,[\int (1+u+u^2+u^3+..)(1-u)^{\alpha-1} du\,] = 1$

i.e ; $C\,[\int (1-u)^{\alpha-2}\, du\,] = 1$

i.e ; $C / (\alpha-1)\,[\,1 - 0\,] = 1$

Hence, $C = \alpha-1$.

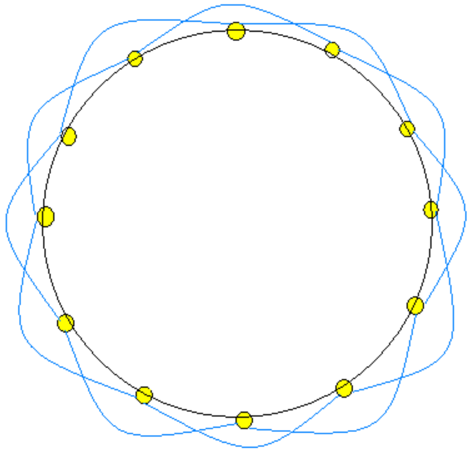
$$
\begin{aligned}
\text{And ,}\quad \sum k p(k) = \sum k C\,\mathrm{B}(k, \alpha) &= C\,[\sum \int k\, u^{k-1}(1-u)^{\alpha-1} du\,] \\
&= C\,[\int (1+2u+3u^2+4u^3+..)(1-u)^{\alpha-1} du\,] \\
&= C\,[\int (1-u)^{\alpha-3}\, du\,] \\
&= C/(\alpha-2) \\
&= (\alpha-1)/(\alpha-2)
\end{aligned}
$$

[because $(1+2u+3u^2+4u^3+..) = 1/(1-u)^2$]

Then, in thermodynamic limits (when k is large and α *is constant*)

$p(k) = (\alpha - 1)\, k^{-\alpha}$ *(because* $\mathrm{B}(k, \alpha) = k^{-\alpha}$*).*

Prob 4.



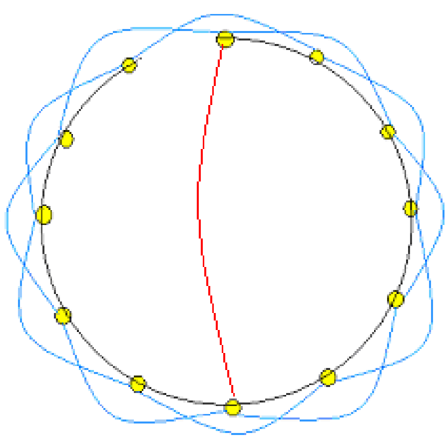
a.) Average distance between all pairs of different nodes

L = n(n+k-2) / 2k(n-1) - For a k-ring

= 12*(12+2-2) / 2*2*(12-1)

= 3.272

b) A single rewiring that minimizes average distance would be to delete a connection between adjacent pair of nodes and then connecting that edge to a diagonally opposite node.



c.) As the rewiring decreases the average distance between all pair of nodes in the graph, the graph tends towards a small-world network. And for a small world network we don't have any exact solution for the value of mean path length (ℓ).

In the limit $p \to 0$, the model is a "large world"— the typical path length tends to L/4k. Small-world behavior, by contrast, is typically characterized by logarithmic scaling log L. which we see for large rewiring probability p, where the model becomes like a random graph. In between these two limits there is presumably some sort of crossover from large to small-world behavior. Barth´el´emy and Amaral conjectured that l satisfies a scaling relation of the form

$$l = \xi \, g(L/\xi)$$

where ξ is a correlation length that depends on p, and
g(x) an unknown but universal scaling function that depends only on system dimension and lattice geometry

One more form of the above equation is

$$L = L/k \; f(Lkp)$$

We would like to be able to calculate the scaling function f(x), but this turns out not to be easy. The calculation is possible, though complicated, for a variant model in which there are no short cuts but random sites are connected to a single central "hub" vertex. But for the normal small-world model no exact solution is known.