Effect of attack on correlated attack

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1 Introduction

A network is said to be *correlated[1]* when the probability that an edge departing from a vertex of degree k arrives at a vertex of degree k' is not independent of the degree of the initial vertex k. In the presence of positive correlations (vertices with large degree tend to connect more preferably with vertices with large degree), the network is said to show *assortative mixing[1]*. On the other hand, negative correlations (highly connected vertices are preferably connected to vertices with low degree) imply the presence of *dissortative mixing[1]*. In correlated superpeer network, assortative mixing occurs if superpeers tend to connect with superpeers. On the other hand, if superpeers tend to connect with peers then we have dissortative mixing.

Here, we present a framework to explain the exact behavior of correlated networks in the face of deterministic attack.

2 Attacks

1. **Deterministic attack**: Superpeer nodes are targeted before attacking any peer[2]. Formally $q_k = 0$ when $k > k_{max}$

$$0 \leq q_k < 1 \text{ when } k = k_{max}$$

$$q_k = 1 \text{ when } k < k_{max}$$

This removes a fraction of nodes from the network with degree $\geq k_{max}$.

2. **Degree dependent attack** : Both peers and superpeers are attacked simultaneously[2], but the probability of superpeers being attacked is much more than that of the peers. Formally the probability of removal of a node having degree k (f_k) is proportional to k γ where $\gamma \geq 0$ is a real number.

Stability metric: The stability of superpeer networks are primarily measured in terms of certain fraction of nodes (f_c) called percolation threshold [3], removal of which disintegrates the network into large number of small, disconnected components.

3 Effect of attack on Un-correlated networks

An attack can be thought of in the following way:

Let p_k be the degree distribution of the network before the attack. The first step in the attack is to select the nodes that are going to be removed. Let us assume that this is performed by means of f_k, where f_k represents the probability for a node of degree k of being removed from the network. Note that the only restriction on f_k is $0 \leq fk \leq 1$. After the node selection, we divide the network into two subsets, one subset contains the surviving nodes (*S)* while the other subset comprises the nodes that are going to be removed (*R)*. The probability Φ of finding an edge in subset *S* that is connected to a node in subset *R* is expressed as[4]:

$$\Phi = \frac{\sum_{i=0}^{\infty} i p_i f_i}{\left(\sum_{k=0}^{\infty} k p_k\right) - {}^{1}/_{N}}$$

The reasoning behind this expression is as follows. The total number of half-edges in the surviving subset, including the *E* links that are going to be removed, is $\sum_{j=0}^{\infty} j\left(Np_j\right)(1 - f_j)$. The probability for a randomly chosen half-edge of being removed is simply $\sum_{i=0}^{\infty} i(Np_i)f_i$ / $[\sum_{k=0}^{\infty} k(Np_k) - 1]$. *E* is the number of half-edges in *S* times this probability, and Φ is obtained by dividing *E* by the number of half-edges in the subset *S*. Notice that the removal of nodes can only lead to a decrease of the degree of a node. Finally, to calculate the degree distribution $p_k^{'}$ after the attack, we still need to estimate the probability p_q^s of finding a nodes with degree q in the surviving subset *S (*before cutting the *E* edges). This fraction takes the simple form[4]

$$p_q^s = \frac{(1 - f_q)p_q}{1 - \sum_{i=0}^{\infty} p_i f_i}$$

Now we are in condition to compute $p_k^{'}$. we obtain the following expression for $p_k^{'}$[4]:

$$p_k^{'} = \sum_{q=k}^{\infty} \binom{q}{k} \Phi^{q-k} (1 - \Phi)^k p_q^s$$

4 Effect of attack on Correlated networks

Modify the expression (derived above) of deformed degree distribution $p_k^{'}$ to make it suitable for degree correlated networks.

Correlation Matrix

The degree-degree correlation information of a network with maximum degree k_M is represented by the correlation matrix *M* as follows

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} & \dots & m_{1k_M} \\ m_{21} & m_{22} & m_{23} & \dots & m_{2k_M} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ m_{k_M 1} & m_{k_M 2} & m_{k_M 3} & \dots & m_{k_M k_M} \end{pmatrix}$$

In this correlation matrix *M*, each element m_{jk} represents the fraction of total edges.

That exists between nodes of degree *j* and nodes of degree *k*. We frame the attack on the network in the same manner as explained in the section 3. The attack on the network divides the graph into two sets of nodes: one set containing the surviving nodes *S* and another set containing the nodes to be removed *R*.

Edge Connectivity (E) between R and S

we have calculated *E* which represents the number of edges running between set *S* and *R*. It is also the number of tips that is going to be removed from the nodes of the set *S*. In case of a degree correlated network, the probability of an edge between a node of degree *i* and a node of degree *j* is given by m_{ij} element of the correlation matrix *M*. Hence instead of calculating *E* we calculate E_j which indicates the number of edges connected between nodes of degree *j* in the set *S* and the nodes of any degree in the set *R*. Hence the total number of edges connected between the set *S* and *R*, that are going to be removed is given by $E = \sum_{j=0}^{k_M} E_j$. The expression for E_j can be formulated in the following way.

The total number of edge tips connected to the *k* degree nodes in set *R* can be expressed as kn_kf_k. Therefore, the number of edge tips connected to the *j* degree nodes of the network whose other end is connected to the *k* degree node of set *R* becomes $m'_{jk}\, kn_k f_k$. The fraction m'_{jk} represents the fraction of edges connecting *j* degree nodes and *k* degree nodes over all the edges in the network with at least one end connected to the *k* degree nodes. The value of m'_{jk} can be computed from the edge correlation matrix M as

$$m'_{jk} = \frac{m_{jk}}{\sum_{j=0}^{\infty} m_{jk}} = \frac{m_{jk}}{kp_k} \sum_i ip_i$$

Where $\sum_{j=0}^{\infty} m_{jk}$ denotes the fraction of edge tips connected to k degree nodes in the network and may be expressed as

$$\sum_{j=0}^{\infty} m_{jk} = \frac{kp_k}{\sum_i ip_i}$$

Now the number of edge tips connected to the *j* degree nodes of set *S* whose other end is connected to the *k* degree node of set *R* becomes $m_{jk}^{'} kn_k f_k (1 - f_j)$. This helps us to derive the total number of edges whose one end is connected to a *j* degree node in set *S* and the other end is connected to any node in the set *R*, which can be expressed as

$$E_j = \sum_{k=0}^{\infty} m_{jk}^{'} kn_k f_k (1 - f_j)$$

Deformed degree distribution

Due to the presence of degree correlation, the probability that a surviving node of set *S* loses one link due to the removal of *E* edges is not constant (as Φ). Moreover, the probability that a survived node loses one link depends upon the degree (*j*) of the survived node. Hence, the probability Φ_j of finding an edge running between a *j* degree node in the surviving set *S* and any node of the other set *R* can be expressed as

$$\Phi_j = \frac{E_j}{jn_j (1 - f_j)}$$

Here Φ_j signifies the probability that a *j* degree node loses one link due the removal of *E* edges.

Finally, the expression of the deformed degree distribution can be expressed in binomial distribution form

$$p_k^{'} = \sum_{q=k}^{\infty} \binom{q}{k} \Phi_q^{q-k} (1 - \Phi_q)^k p_q^s$$

where p_q^s is the probability of finding a node with degree *q* in the surviving subset *S* (before removal of the *E* edges).

5 References

1. “Class of correlated random networks with hidden variables”, Maria´n Bogun and Romualdo Pastor-Satorras
2. “How stable are large superpeer networks against attack?” Bivas Mitra, Sujoy Ghose, Niloy Ganguly
3. D. S. Callaway , M. E. J. Newman, S. H. Strogatz, D. J. Watts : “Network Robustness and Fragility: Percolation on Random graphs”, Vol. 85, No. 21 Physical Review Letters, 2000.
4. “Generalized theory for node disruption in finite-size complex networks”, Bivas Mitra, Niloy Ganguly, Sujoy Ghose, and Fernando Peruani.