

# Digraphs and types of relations

Prepared by: **Ashish Kumar Yadav(07CS3028)**

*Teacher:* Prof. Niloy Ganguly  
Department of Computer Science and Engineering  
IIT Kharagpur

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# 1 The digraph of a relation

If  $A$  is a finite set and  $R$  a relation on  $A$ , we can also represent  $R$  pictorially as follows:  
 Draw a small circle for each element of  $A$  and label the circle with the corresponding element of  $A$ . These circles are called the vertices. Draw an arrow, called an **edge**, from vertex  $a_i$  to  $a_j$  if and only if  $a_i R a_j$ .

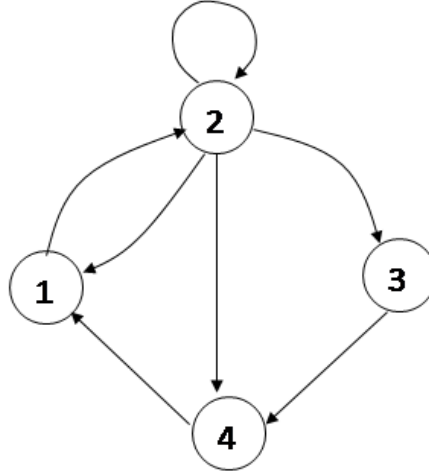
The resulting pictorial representation of  $R$  is called a **directed graph** or **digraph** of  $R$ .

Let

$$A = \{1, 2, 3, 4\}$$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 2)\}$$

Then the digraph is as shown:



## 1.1 In-degree and Out-degree

If  $R$  is a relation on a set  $A$  and  $a \in A$ , then the **in-degree** of  $a$  is the number of  $b \in A$  such that  $(b, a) \in R$ . The **out-degree** of  $a$  is the number of  $b \in A$  such that  $(a, b) \in R$ .

What this means, in the term of the digraph of  $R$ , is that the *in-degree* of a vertex is the number of edges terminating at that vertex. The *out-degree* of a vertex is the number of edges leaving that vertex.

Consider the digraph shown earlier:

| Vertex     | 1 | 2 | 3 | 4 |
|------------|---|---|---|---|
| In-degree  | 3 | 2 | 1 | 2 |
| Out-degree | 2 | 4 | 1 | 1 |

If  $R$  is a relation on a set  $A$ , and  $B$  is a subset of  $A$ , the **restriction of  $R$  to  $B$**  is  $R \cap (B \times B)$ .  
 Let  $A = \{a, b, c, d, e, f\}$ ,  $R = \{(a, a), (a, c), (b, c), (a, e), (b, e), (c, e)\}$  and  $B = \{a, b, c\}$ . Then

$$B \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

and the restriction of  $R$  to  $B$  is  $\{(a, a), (a, c), (b, c)\}$ .

## 2 Paths in relation

Suppose that  $R$  is a relation on a set  $A$ . A **path of length  $n$**  in  $R$  from  $a$  to  $b$  is a finite sequence  $\pi : a, x_1, x_2, \dots, x_{n-1}, b$ , beginning with  $a$  and ending  $b$ , such that

$$aRx_1, x_1Rx_2, \dots, x_{n-1}Rb.$$

A path that begins and ends at the same vertex is called a **cycle**.

If we have  $aRb$  and  $bRc$  then there exists a path of length two from  $a$  to  $c$  and it is represented by  $aR^2c$ .

We have  $M_{R^2} = M_R \odot M_R$

where  $M_R$  is the matrix for relation  $R$  and  $M_{R^2}$  for  $R^2$ .

We have the following relations:

$R^n$  means a path of length  $n$  exists in  $R$

$R^\infty$  means there is some path in  $R$ .

$$R^\infty = R \cup R^2 \cup R^3 \cup \dots \cup R^{n-1}$$

$$M_{R^\infty} = M_R \vee M_{R^2} \vee M_{R^3} \vee \dots$$

The **reachability** relation  $R^*$  of a relation  $R$  on a set  $A$  that has  $n$  elements is defined as follows:  $xR^*y$  means that  $x = y$  or  $xR^\infty y$ . It is seen that  $M_{R^*} = M_{R^\infty} \vee I_n$ , where  $I_n$  is the identity matrix.

## 3 Types of Relations

### 3.1 Reflexive and Irreflexive Relations

A relation  $R$  on a set  $A$  is **reflexive** if  $(a, a) \in R$ , that is, if  $aRa$  for all  $a \in A$ . A relation  $R$  on a set  $A$  is **irreflexive** if  $(a, a) \notin R$  for every  $a \in A$ .

**Examples:**

$A = \{1, 2, 3\}$ ,  $R = \{(1, 1), (1, 2)\}$  in this  $R$  is neither reflexive nor irreflexive.

$A = \{1\}$ ,  $R = \emptyset$  in this  $R$  is irreflexive.

We can identify a reflexive or irreflexive relation by its matrix as follows. The matrix of a reflexive relation must have all 1's on its main diagonal, while the matrix of an irreflexive relation must have all 0's on its main diagonal.

Similarly, we can characterize the digraph of a reflexive or irreflexive relation as follows. A reflexive relation has a cycle of length 1 at every vertex, while an irreflexive relation has no cycles of length 1.

### 3.2 Symmetric, Asymmetric and Antisymmetric Relations

A relation  $R$  on set  $A$  is called

**Symmetric** if whenever  $aRb$  then  $bRa$ ,

**Asymmetric** if whenever  $(a, b) \in R$  then  $(b, a) \notin R$ ,

**Antisymmetric** if whenever  $aRb$  and  $bRa$  then  $a = b$ .

The matrix  $M_R = [m_{ij}]$  of a

(a) Symmetric relation satisfies the property that if  $m_{ij} = 1$ , then  $m_{ji} = 1$ .

(b) Asymmetric relation satisfies the property that if  $m_{ij} = 1$ , then  $m_{ji} = 0$ .

(c) Antisymmetric relation satisfies the property that if  $i \neq j$ , then  $m_{ij} = 0$  or  $m_{ji} = 0$ .

We now consider the digraphs of these three types of relations.

The digraph of a symmetric relation has a property that if there exists an edge from vertex  $i$  to vertex  $j$ , then there is an edge from vertex  $j$  to vertex  $i$ .

If  $R$  is an asymmetric relation, then digraph of  $R$  cannot simultaneously have an edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$ . Thus there can be no cycles of length 1, and all edges are "one-way streets".

If  $R$  is an antisymmetric relation, then for different vertices  $i$  and  $j$  there cannot be edge from vertex  $i$  to vertex  $j$  and an edge from vertex  $j$  to vertex  $i$ . When  $i = j$ , no condition is imposed. Thus there may be cycles of length 1, but again all edges are "one-way".

### 3.3 Transitive Relations

We say that a relation  $R$  on a set  $A$  is **transitive** if whenever  $aRb$  and  $bRc$ , then  $aRc$ . It is often convenient to say what it means for a relation to be not transitive. A relation  $R$  on  $A$  is not transitive if there exist  $a, b$ , and  $c$  in  $A$  so that  $aRb$  and  $bRc$ , but  $(a, c) \notin R$ . If such  $a, b$ , and  $c$  do not exist, then  $R$  is transitive.

**Example:**

Let  $A = \mathbb{Z}$ , the set of integers, and let  $R$  be the relation *less than*. To see whether  $R$  is transitive, we assume that  $aRb$  and  $bRc$ . Thus  $a < b$  and  $b < c$ . It then follows that  $a < c$ , so  $aRc$ . Hence  $R$  is transitive.

A relation  $R$  is transitive if and only if its matrix  $M_R = [m_{ij}]$  has the property:

if  $m_{ij} = 1$  and  $m_{jk} = 1$ , then  $m_{ik} = 1$ .

In more simple words, if  $(M_R)_{\odot}^2 = M_R$ , then  $R$  is transitive. The converse is not true.

**Example:**

Let  $A = \{1, 2, 3\}$  and let  $R$  be the relation on  $A$  whose matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

By direct computation,  $(M_R)_{\odot}^2 = M_R$ ; therefore,  $R$  is transitive.

How can we identify a transitive relation from its digraph?

For a transitive relation, if  $a$  and  $c$  are connected by a path of length 2 in  $R$ , then they must be connected by a path of length 1.