Discrete Structures Mid-Semester Solutions

1 Question I

1(a): Let $m = p_1^{a_1} p_2^{a_2} p_3^{a_3} ....p_n^{a_n}$ and $n = q_1^{b_1} q_2^{b_2} q_3^{b_3} ....q_n^{b_n}$ where all p_i's and q_i's are prime and distinct. If mn is a perfect square, then, $p_1^{a_1} p_2^{a_2} ....p_n^{a_n} q_1^{b_1} q_2^{b_2} ....q_n^{b_n}$ is a perfect square. Also as all of p_i's and q_i's are distinct, so all the powers a_i's and b_i's must be even. So m and n both are perfect squares.

1(b): Let $X = AB$, Premultiplying both sides by $B^{-1}A^{-1}$, we get, $B^{-1}A^{-1}T = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$ Similarily, we can show that by postmultiplying $B^{-1}A^{-1}$ we get I. So, $B^{-1}A^{-1} = X^{-1}$, but $X = AB$, so $B^{-1}A^{-1} = (AB)^{-1}$

1(c):

(i)

$$T_n = T_{n-1}^2 + T_{n-2}^2 + T_{n-3}^2 + ... + T_1^2 + T_0^2$$

$$T_{n-1} = T_{n-2}^2 + T_{n-3}^2 + T_{n-4}^2 + ... + T_1^2 + T_0^2$$

Subtracting,

$$T_n - T_{n-1} = T_{n-1}^2$$

or,

$$T_n = T_{n-1}(T_{n-1} + 1)$$

(ii) We prove by Induction on n. $T_0 = 2 = 2^{2^0}$. So we assume $n \geq 1$ and that $T_{n-1} \geq 2^{2^{n-1}}$. The recurrence implies $T_n \geq T_{n-1} \geq (2^{2^{n-1}})^2 = 2^{2^n}$.

(iii) We prove by Induction on n. $T_0 = 2 = 2^{3^0}$ and $T_1 = T_0^2 = 4 \leq 2^{3^1}$. So we assume that $n \geq 2$ and $T_{n-1} \leq 2^{3^{n-1}}$. By part (i) we have, $T_n = T_{n-1}(T_{n-1} + 1)$. And since $x + 1 \leq x^2$ for all $x \geq 2$, we have

$$T_n \leq T_{n-1}^3 \leq (2^{3^{n-1}})^3 = 2^{3^n}$$

1(d): We prove by contrapositive. Thus we need to prove that if not both p and q are odd, then their product pq is not odd (i.e. it must be even). Let $p = 2m$ (even) and $q = (2m + 1)$ (odd) for some $m,\ n \in Z$. Now the product $pq = 2m(2n + 1) = 4mn + 2m = 2(2mn + m)$, which is even. Similarly for the case when both p and q are even, pq is again even. Hence, if not both p and q are odd, then their product pq is not odd.

1(e): Consider the function f defined on X by the rule that for each member x of $X : f(x) =$ Number of friends of x in X. If $\mid X \mid = m$,then the possible values of $f(x)$ are $0, 1, 2, ...m - 1$. Since friends of x can be members of X except x, so f is a function from set X to set $Y = \{0, 1, 2, ...m-1\}$. At this point we cannot apply Pigeon Hole Principle as the cardinality of both the sets is m. If there is a person c who has $m - 1$ friends then everyone else is a friend of c and no one has 0 friends. So the numbers $m - 1$ and 0 can not be both the members of set Y. So f is a function from a set with m members to a set with $m - 1$ (or lesser) members and PHP tells us that f cannot be an injective function. So, at least 2 people must have same number of friends.

1(f): $Y = \{1, 3^{1/2}, 3, 3^{3/2}, ..., 3^{19/2}, 3^{10}\}$
We can observe that the given numbers are in GP. We want the number of ways of selecting 2 numbers such that their product is not less than 3^{10}. In other we have to choose 2 numbers from the set {0, 1, 2, 3, ... ,20} such that their sum is greater than 20. We have the following cases:

0 : (20)
1 : (20,19)
2 : (20,19,18)
..
..
9 : (20,19,......,11)
10: (20,19,......,11)
11: (20,19,....,12)
..
..
19: (20)

So, the total number of ways of selecting two numbers from Y such that product is greater than or equal to 3^{10} is $(1 + 2 + 3 + ... + 10) + (1 + 2 + 3 + ... + 10) = 110$.

1(g): Let $S_{n,3}$ denote the total number of bit strings of length n that have three consecutive $0s$. Now we have two possibilities, if 1 occurs on the first place or 0 occurs in the first place. If 1 occurs, then the problem reduces to finding the number of bit strings of length $n - 1$ that have three consecutive 0s i.e. $S_{n-1,3}$ and if 0 occurs in the first place then we look for the 2nd digit. Similarly if 1 occurs in the 2nd digit as well, then we have to solve the relation $S_{n-2,3}$, and if again 0 occurs we see the 3rd digit, if 1 occurs here then we solve $S_{n-3,3}$ otherwise the remaining $n - 3$ places can be either filled with 1 or 0, giving a total number of ways as 2^{n-3}. Hence the recurrence relation becomes

$$S_{n,3} = S_{n-1,3} + S_{n-2,3} + S_{n3,3} + 2^{n-3}$$

$$S_{2,3} = 0, S_{1,3} = 0, S_{0,3} = 0$$

From the above recurrence relation, we get

$$S_{3,3} = 1,\ S_{4,3} = 3,\ S_{5,3} = 8,\ S_{6,3} = 20,\ S_{7,3} = 47$$

1(h): Since $|A_1| = |A_2| = |A_3|$, and $|A_1| + |A_2| + |A_3| = 24$, we have $|A_1| = |A_2| = |A_3| = 8$. Within each partition, each element is related with every element. Hence $|q| = 3 \times (8 \times 8) = 192$.

2 Question II

2(a): Applying Warshall's Algorithm of Transitive Closure, We get the smallest equivalence relation containing R and S as

$$(R \cup S)^{\infty} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$$

The corresponding partition of A is then{{1, 2},{3, 4, 5}}.

2(b):

$$R_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

2(c): Suppose $A = \{a, b, c\}$ and $R = \{(a, b), (b, a), (a, a), (b, b)\}$. In this example, R is symmetric and transitive, but not reflexive as (c, c) is not included. The proof is wrong because for transitivity requires all 3 elements to be different, but in this case this does not hold true.

2(d): $b_n = 2(b_{n-1} - 1)$
$= 2(2b_{n-2} - 1) - 1$
$= 4b_{n-1} - 2 - 1$
$= ....$
$= 2^{k+1}b_{n-k} - 2^{k+1} + 1$
Taking $k = n - 1$, we get,
$b_n = 7 \times 2^n - 2^n + 1$

3 Question III

3(a): If a and b are in the set A, then $aR^{\infty}b$ if and only if there is a path in R from a to b. Now R^{∞} is certainly transitive. To show that R^{∞} is the smallest transitive relation containing R, we must show that if S is any transitive relation on A and $R \subseteq S$, then $R^{\infty} \subseteq S$. We know that if S is transitive, then $S^n \subseteq S$ for all n, ie if a and b are connected by a path of length n, then aS^nb. It follows that $S^{\infty} \subseteq S$. It is also true that if $R \subseteq S$, then $R^{\infty} \subseteq S^{\infty}$, since any path in R is also in S. Putting these facts together, we see that if $R \subseteq S$ and S is transitive on A, then $R^{\infty} \subseteq S^{\infty} \subseteq S$. This means that R^{∞} is the smallest of all transitive relations on A that contain R.

3(b): First suppose that $R(a) = R(b)$. Since R is reflexive, $b \in R(b)$; therefore, $b \in R(a)$, so $a\ R\ b$.

Conversely, suppose that $a\ R\ b$. Now $b \in R(a)$, and $a \in R(b)$ (Since R is symmetric).

We must show that $R(a) = R(b)$. We choose an element $x \in R(b)$. Since R is transitive, and using statements above, we get $x \in R(a)$. Therefore $R(b) \subseteq R(a)$. Similarily, the reverse can also be proved. Hence $R(a) = R(b)$.

Naveen Kumar Sharma

07CS3027