Isomorphism and homomorphism

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ISOMORPHISM:-

Let (S,*) and (T,*') be two semigroups.A function $f:S \rightarrow T$ is called an isomorphism from (S,*) to (T,*') if it a one-to-one correspondance from $S \rightarrow T$ and if

$$\boxed{f(a*b)=f(a)*'f(b)}$$

for all a and b in S

if f is an isomorphism from (S,*) to (T,*') then since f is one-to-one correspondance,it follows that f^{-1} exists and is a one-to-one correspondance from T to S and f^{-1} is an isomorphism from (T,*')to (S,*)

How to show that two groups are isomorphic

1.Define a function $f:S \rightarrow T$ with Dom(f)=S
2.Show that f is one-to-one
3.Show that f is onto
4.Show that $f(a*b)=f(a)*'f(b)$

Question:-Show that (Z,+) and (T,+) are isomorphic.Where T is a set of even number

Solution:-
Steps:

1.

We define the function $f:Z \rightarrow T$ by $f(a)=2a$

2.

Let $f(a)=f(a')$

then $2a=2a$

$a=a'$

Hence f is one-to-one

3.

Suppose b is any even integer .Then $a=b/2$ where $a \in Z$

and $f(a)=f(b/2)=2(b/2)=b$

so f is onto

4.

$f(a+b)=2(a+b)=2a+2b=f(a)+f(b)$

Hence $(Z,+)$ and $(T,+)$ are isomorphic

Theorem:-Let $(S,*)$ and $(T,*')$ be monoid with identity e and e' respectively if $f:S \rightarrow T$ be an isomorphism then $f(e)=e'$

Proof:-

Let $a \in A$ and $b \in B$

$a=a*e$ and $b=b*e'$ since e and e'are an identity

and $b=f(a)$

$=f(a*e)$

$=f(a)*'f(e)$

$=b*'f(e)$

Therefore $f(e)=e'$

Example-

Are (T,x) and (Z,x) isomorphic? Where "x " is multiplication

Ans-

No because (Z,x) has identity as 1 but (T,x) doesn't have.

Homomorphism:-

Let (S,*) and (T,*) be two semigroups.A function $f:S \rightarrow T$ is called an homorphism from (S,*) to (T,*') if

$$\boxed{f(a*b)=f(a)*'f(b)}$$

Theorem:-Let (S,*) and (T,*') be monoid with identity e and e' respectively if $f:S \rightarrow T$ be an homorphic function then

$$f(e)=e'$$

Example-Let A={0,1} and consider the semigroups (A*,c) and

Theorem:-Let f be a homomorphism from a semigroup (S,*) to a semigroup (T,*') .If S' is a subsemigroup of (S,*),then

$$f(S')=\{t \in T \mid t=f(s) \text{ for some } s \in S'\}$$

the image of S' under f, is a subsemigroup of (T,*')

Proof:-

If t=f(s) and t'=f(s')

Then t*'t'=f(s)*'f(s')=f(s*s')=f(s'')

Where s''=s*s' $\in$ S'.

Hence t*t' ∈ f(S')
Thus f(S') is closed under operation *'
Since associativety property holds in T,it holds in f(S'),so f(S') is a subsemigroup of (T.*')

Theorem:-If f is a homomorphism from commutative semigroup (S,*) onto a semigroup (T,*').then (T,*') is also commutative

Proof:-
Let t and t' be any element of T .Then there exist s and s'in S with t=f(s) and t'=f(s')
Therefore

$$t*'t'=f(s)*'f(s')=f(s*s')=f(s'*s)=f(s')*'f(s) \\ =t'*t$$

Hence (T,*') is also commutative