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Discrete Mathematics – Lecture 34
Special Lattices

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Contents

1 Intoduction

We will study different type of Lattices and their properties namely distributiveness

2 Bounded Lattice

A lattice L is bounded if it has a greatest element I and a least element 0.

2.1 Examples

1. The lattice Z+ under the partial order of divisibility, is not bounded as it has no greatest element although it has a least element i.e 1.
2. The lattice P(S) of all subsets of a set is bounded.Its greatest element is S itself and least element is ϕ
   $0 \leq a \leq 1$
   $a \vee 0 \quad a \wedge 0=0$
   $a \vee I=I \quad a \wedge I=a$

3 Theorem

Let L=a1,a2,.....,an be a finite lattice.Then L is bounded

1. By construction
   The greatest element of L is a1 $\vee$ a2 $\vee$ .... $\vee$ an
   and 0=a1 $\wedge$ a2 $\wedge$ ...... $\wedge$ an.
   Hence L is bounded.

4 Distributive properties

1. A lattice is called distributive if for any element a,b and c in L we have following distributive properties.
2. a $\wedge$ (b $\vee$ c)=(a $\wedge$ b) $\vee$ (a $\wedge$ c)
3. a $\vee$ (b $\wedge$ c)=(a $\vee$ b) $\wedge$ (a $\vee$ c)
4. If L is not distributive we call it non distributive
5. Example–For a set S,P(S) is distributive since $\bigcup$ and $\bigcap$ are distributive operations
6. So the point here is how to decide whehter a particular lattice is non distributive or not
   We state here a Theorem without proof for this.

5 Theorem

A Lattice is nondistributive iff it contains a sublattice that is isomorphic to one of these lattices in fig2 and fig3

In Fig 2
a $\wedge$ (b $\vee$ c)=(a $\wedge$ b) $\vee$ (a $\wedge$ c)
a $\wedge$ I=b $\vee$ 0
a $\neq$ b

In Fig 3
a $\wedge$ (b $\vee$ c)=(a $\wedge$ b) $\vee$ (a $\wedge$ c)
a $\wedge$ I=0 $\vee$ 0
a $\neq$ 0

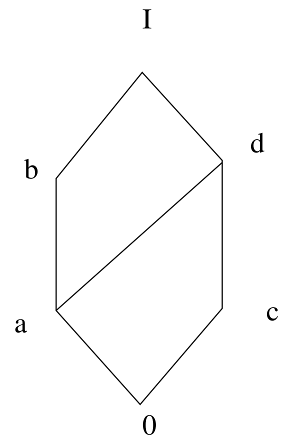

Figure 1: Distributive Lattices

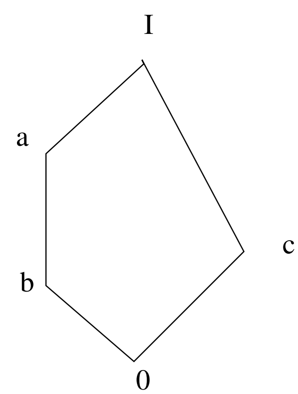

Figure 2: Non Distributive Lattices

Planar Graphs These are graphs drawn where no edge intersect the other. Firsy non planar graph is K5 ie clique of size 5.

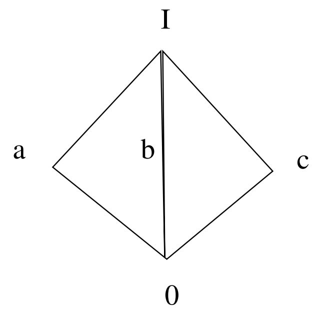

Figure 3: Non Distributive Lattices

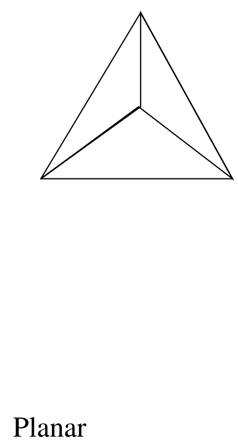

Figure 4: Planar Graph

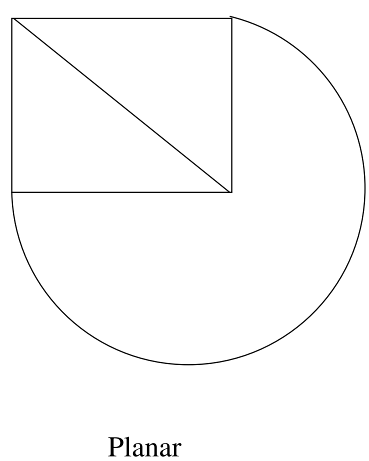

Figure 5: Planar Graph

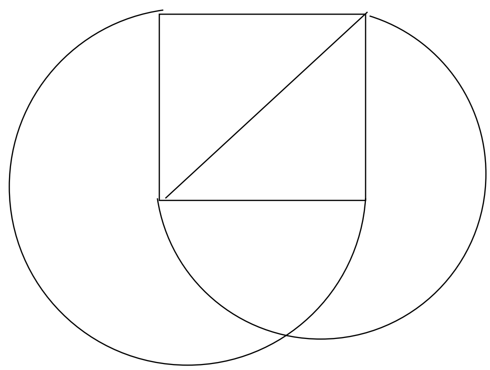

Figure 6: Non Planar Graph

6 Complement

Let L be a bounded lattice with greatest element I and least element 0 and let a $\in$ L.An element a' $\in$ L is called a complement of a if a $\wedge$ a'=I a $\vee$ a'=0
observe that
0'=I I'=0

6.1 Examples

1. The lattice L=P(S) is such that every element has a complement,since is A $\in$ L,then its complement A' has the properties A $\vee$ A'=S and A $\wedge$ A'= ϕ.

7 Theorem

7.1 For a bounded lattice uniqness holds true for complement.

Let a' and a" be complemnets of a $\in$ L.Then
a $\wedge$ a'=I and a $\vee$ a'=0
a $\wedge$ a"=I and a $\vee$ a"=0
using the distributive laws,we obtain a'=a' $\vee$ 0=a' $\vee$ (a $\wedge$ a")
=(a' $\vee$ a) $\wedge$ (a' $\vee$ a")
=I $\wedge$ (a' $\vee$ a")=a' $\vee$ a".
Also
a"=a" $\vee$ 0=a" $\vee$ (a $\wedge$ a')

$=(a'' \vee a) \wedge (a'' \vee a')$
$=I \wedge (a' \vee a'')=a' \vee a''$.
Hence $a'=a''$.

A lattice is called complemented if it is bounded and if every element in it has a complement