Discrete Structures(CS21001)

Prof. Niloy Ganguly

Lecture on –

LATTICES

Prepared by-

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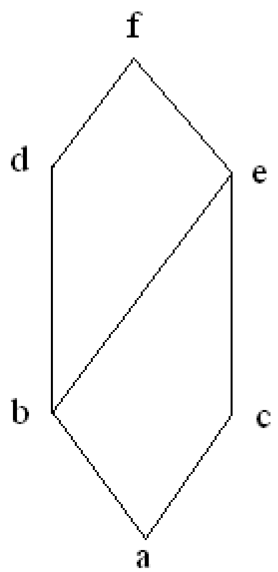
Topics Covered-

1. General introduction of Lattice
2. Properties of Lattices.
3. Types of Lattices.
4. Important Theorems.

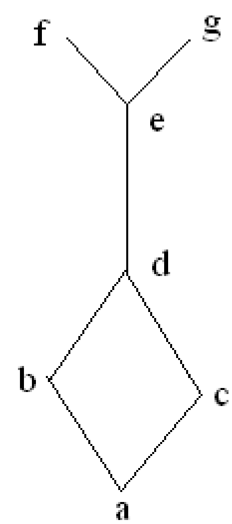
I. Lattice

A lattice is a poset in (L,≤) in which every subset {a,b} consisiting of two elements has a least upper bound and a greatest lower bound.
LUB({a,b}) is denoted by a v b and is called the join of a and b.
GLB({a,b}) is denoted by a Λ b and is called the meet of a and b.

Eg.



(a)



(b)

a) is a lattice.
b) Is not a lattice because f v g does not exist.

Theorem:

If (L1,≤) and (L2,≤) are lattices then (L,≤) is a lattice where L = L1xL2 and the partial order ≤ of L is the product partial order.

II. Properties of Lattices

1. Idempotent Properties

   a) $a \vee a = a$
   b) $a \wedge a = a$

2. Commutative Properties

   a) $a \vee b = b \vee a$
   b) $a \wedge b = b \wedge a$

3. Associative Properties

   a) $a \vee (b \vee c) = (a \vee b) \vee c$
   b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

4. Absorption Properties

   a) $a \vee (a \wedge b) = a$
   b) $a \wedge (a \vee b) = a$

<u>Theorem:</u>

a) $a \vee b = b$ iff $a \leq b$
b) $a \wedge b = a$ iff $a \leq b$
c) $a \wedge b = a$ iff $a \vee b = b$

Theorem:

1. If $a \leq b$ then
   a) $a \vee c \leq b \vee c$
   b) $a \wedge c \leq b \wedge c$
2. $a \leq c$ and $b \leq c$ iff $a \vee b \leq c$
3. $c \leq a$ and $c \leq b$ iff $c \leq a \wedge b$
4. if $a \leq b$ and $c \leq d$ then
   a) $a \vee c \leq b \vee d$
   b) $a \wedge c \leq b \wedge d$

III. Types of Lattices

1. Isomorphic Lattices

   If f: L1 -> L2 is an isomorphism from the poset (L1,≤1) to the poset (L2,≤2) then L1 is a lattice iff L2 is a lattice.

   If a and b are elements of L1 then
   $f(a \wedge b) = f(a) \wedge f(b)$ and
   $f(a \vee b) = f(a) \vee f(b)$

   If two lattices are isomorphic as posets we say they are isomorphic lattices.

2. Bounded Lattice

   A lattice L issaid to be bounded if it has a greatest element I and a least element 0.

3. Complemented Lattice

   A lattice L is said to be complemented if it is bounded and if every element in L has a complement.

Theorem:

Let L = {a1,a2,a3,a4…..an} be a finite lattice. Then L is bounded.

Theorem:

Let L be a bounded lattice with greates element I and least element 0 and let a belong to L. an element a' belong to L is a complement of a if

$$a \vee a' = I \text{ and } a \wedge a' = 0$$

Theorem:

Let L be a bounded distributive lattice. If complement exists it is unique.