Lecture 25
FUNCTIONS

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1 Definition

Let A and B be non-empty sets. A **function** f from A to B, which is denoted $f : A \to B$, is a relation from A to B such that for all $a \in Dom(f)$, $f(a)$, the f-relative set of a, contains just one element of B.If a is not in $Dom(f)$, then $f(a) = \phi$. If $f(a) = b$, then we identify the set b with the element b and write $f(a) = b$. Functions are also called **mappings** or **transformations**, since they can be geometrically viewed as rules that assign to each element $a \in A$ the unique element $f(a) \in B$. The element a is called an **argument** of the function f, and $f(a)$ is called the **value** of the function for the argument a and is also referred to as the **image** of a under f.

2 Special Types of Functions

Let f be a function from A to B. Then we can say the following:

$\triangleright f$ is **everywhere defined** if $Dom(f) = A$

$\triangleright f$ is **onto** if $Ran(f) = B$

$\triangleright f$ is **one to one** if we cannot have $f(a) = f(a')$ for two distinct elements a and a' of A. Equivalently we can state the definition as: if $f(a) = f(a')$, then $a = a'$.

If $f : A \to B$ is a one-to-one function, then f assigns to each element a of $Dom(f)$ an element $b = f(a)$ of $Ran(f)$. Every b in $Ran(f)$ is matched, in this way, with one and only one element of $Dom(f)$. For this reason, such an f is often called a **bijection** between $Dom(f)$ and $Ran(f)$. If f is also everywhere defined and onto, then f is called a **one-to-one correspondence between** A **and** B.

Example

Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2\}$, and $D = \{d_1, d_2, d_3, d_4\}$. Consider the following four functions, from A to B, A to D, B to C, and D to B, respectively.

(a) $f_1 = \{(a_1, b_2), (a_2, b_3), (a_3, b_1)\}$
(b) $f_2 = \{(a_1, d_2), (a_2, d_1), (a_3, d_4)\}$
(c) $f_3 = \{(b_1, c_2), (b_2, c_2), (b_3, c_1)\}$
(d) $f_4 = \{(d_1, b_1), (d_2, b_2), (d_3, b_1)\}$

Determine whether each function is one to one, whether each function is onto, and whether each function is everywhere defined.

Solution

(a) f_1 is everywhere defined, one to one, and onto.
(b) f_2 is everywhere defined and one to one, but not onto.
(c) f_3 is everywhere defined and onto, but not one to one.
(d) f_4 is not everywhere defined, not one to one, and not onto.

3 Invertible Functions

A function $f : A \to B$ is said to be invertible if its inverse relation, f^{-1}, is also a function. All functions are not necessarily invertible. The theorem below states the conditions to be satisfied for f to be invertible.

Theorem Let $f : A \to B$ be a function.

(a) Then f^{-1} is a function from B to A if and only if f is one to one.

If f^{-1} is a function, then

(b) the function f^{-1} is also one to one.
(c) f^{-1} is everywhere defined if and only if f is onto.
(d) f^{-1} is onto if and only if f is everywhere defined.

Proof

(a) We prove the following equivalent statement:

f^{-1} is not a function if and only if f is not one to one.

We suppose first that f^{-1} is not a function. Then, for some b in B, $f^{-1}(b)$ must contain at least two distinct elements, a_1 and a_2. Then $f(a_1) = b = f(a_2)$, so f is not one to one.

Conversely suppose that f is not one to one. Then $f(a_1) = f(a_2) = b$ for two distinct elements a_1 and a_2 of A. Thus $f^{-1}(b)$ contains both a_1 and a_2, so f^{-1} cannot be a function.

(b) Since $(f^{-1})^{-1}$ is the function f, part (a) shows that f^{-1} is one to one.

(c) We know that $Dom(f^{-1}) = Ran(f)$. Thus $B = Dom(f^{-1})$ if and only if $B = Ran(f)$. In other words f^{-1} is everywhere defined if and only if f is onto.

(d) Since $Ran(f^{-1}) = Dom(f)$, $A = Dom(f)$ if and only if $A = Ran(f^{-1})$. That is, f is everywhere defined if and only if f^{-1} is onto.