Lecture 23: Composition of Relations, Transitive Closure and Warshall's Algorithm

Rasha Eqbal
07CS1039
Teacher: Prof. Niloy Ganguly
Department of Computer Science and Engineering
IIT Kharagpur

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1 Composition of Relations

In this section we will study what is meant by composition of relations and how it can be obtained.

Suppose that we have three sets A, B and C; a relation R defined from A to B, and a relation S defined from B to C. We can now define a new relation known as the *composition of* R *and* S, written as $S \circ R$. This new relation is defined as follows. If a is an element in A and c is an element in C, then $a(S \circ R)c$ if and only if there exists some element b in B, such that aRb and bSc. This means that we have a relation $S \circ R$ from a to c, if and only if we can reach from a to c in two steps; i.e. from a to b related by R and from b to c related by S. In this manner relation $S \circ R$ can be interpreted as R followed by S, since this is the order in which the two relations need to be considered, first R then S.

† **Example 1:**

Let us try to understand this better through an example.
Let $A = \{1, 2, 3, 4\}$, $R = \{(1, 2), (1, 1), (1, 3), (2, 4), (3, 2)\}$,
and $S = \{(1, 4), (1, 3), (2, 3), (3, 1), (4, 1)\}$.
Find $S \circ R$.

Solution:
Here we see that (1, 2) $\in R$ and (2, 3) $\in S$. This gives us (1, 3) $\in S \circ R$.
Similarly we can proceed with the others:

- (1, 1) $\in R$ and (1, 4) $\in S \Rightarrow$ (1, 4) $\in S \circ R$
- (1, 1) $\in R$ and (1, 3) $\in S \Rightarrow$ (1, 3) $\in S \circ R$
- (1, 3) $\in R$ and (3, 1) $\in S \Rightarrow$ (1, 1) $\in S \circ R$
- (2, 4) $\in R$ and (4, 1) $\in S \Rightarrow$ (2, 1) $\in S \circ R$
- (3, 2) $\in R$ and (2, 3) $\in S \Rightarrow$ (3, 3) $\in S \circ R$

$\Rightarrow S \circ R = \{(1, 3), (1, 4), (1, 1), (2, 1), (3, 3)\}$.

§ **Theorem 1**

Let R be a relation from A to B and let S be a relation from B to C. Then, if A_1 is any subset of A, we have:

$$(S \circ R)(A_1) = S(R(A_1))$$

Proof

If an element $z \in C$ is in $(S \circ R)(A_1)$, then $x(S \circ R)z$ for some $x \in A_1$. By the definition of composition, this means that xRy and ySz for some $y \in B$.

So now we have $z \in S(y)$ and $y \in R(x) \Rightarrow z \in S(R(x))$.

Since $\{x\} \subseteq A_1$, we can also say that $S(R(x)) \subseteq S(R(A_1))$. Now, since $z \in S(R(x))$, therefore $z \in S(R(A_1))$ also. This means that $(S \circ R)(A_1) \subseteq S(R(A_1))$.

Conversely, suppose $z \in S(R(A_1))$. Then $z \in S(y)$ for some $y \in R(A_1)$. Similarly, $y \in R(x)$ for some $x \in A_1$.

This means that xRy and ySz. So from the definition of composition we can say $x(S \circ R)z$. Thus $z \in (S \circ R)(x)$. Since $\{x\} \subseteq A_1$, we can say that $(S \circ R)(x) \subseteq (S \circ R)(A_1)$. Hence z also belongs to $(S \circ R)(A_1)$. So $S(R(A_1)) \subseteq (S \circ R)(A_1)$.

Since $(S \circ R)(A_1) \subseteq S(R(A_1))$ and $S(R(A_1)) \subseteq (S \circ R)(A_1)$, we can say that

$$(S \circ R)(A_1) = S(R(A_1)).$$

This proves the theorem.

† **Example 2:**

Let $A = \{a, b, c\}$ and let R and S be relations on A whose matrices are:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Find $S \circ R$.

Solution:

We see from the matrices that:

- $(a, a) \in R$ and $(a, a) \in S \Rightarrow (a, a) \in S \circ R$
- $(a, c) \in R$ and $(c, a) \in S \Rightarrow (a, a) \in S \circ R$
- $(a, c) \in R$ and $(c, c) \in S \Rightarrow (a, c) \in S \circ R$

It is easily seen that $(a, b) \notin S \circ R$ since, if we had $(a, x) \in R$ and $(x, b) \in S$, then from Matrix M_R we know that x would have to be either a or c; but from matrix M_S we know that neither (a, b) nor (c, b) is an element of S.

Hence we see that the first row of $M_{S \circ R}$ is 1 0 1.

Proceeding in a similar manner we get:

- $(b, a) \in R$ and $(a, a) \in S \Rightarrow (b, a) \in S \circ R$

- $(b, b) \in R$ and $(b, b) \in S \Rightarrow (b, b) \in S \circ R$
- $(b, b) \in R$ and $(b, c) \in S \Rightarrow (b, c) \in S \circ R$
- $(b, c) \in R$ and $(c, a) \in S \Rightarrow (b, a) \in S \circ R$
- $(b, c) \in R$ and $(c, c) \in S \Rightarrow (b, c) \in S \circ R$

Hence the second row of $M_{S \circ R}$ is 1 1 1.

- $(c, b) \in R$ and $(b, b) \in S \Rightarrow (c, b) \in S \circ R$
- $(c, b) \in R$ and $(b, c) \in S \Rightarrow (c, c) \in S \circ R$

Hence the third row of $M_{S \circ R}$ is 0 1 1.
Therefore the composition matrix is

$$M_{S \circ R} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now we shall deduce an important and useful result.

Let us consider three sets, $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_p\}$ and $C = \{c_1, \dots, c_m\}$; and relation R defined from A to B, and S defined from B to C. The Boolean matrices M_R and M_S are of sizes $n \times p$ and $p \times m$ respectively. Let us represent M_R as $[r_{ij}]$, M_S as $[s_{ij}]$ and $M_{S \circ R}$ as $[t_{ij}]$. Now $t_{ij} = 1 \Leftrightarrow (a_i, c_j) \in S \circ R$, which means that for some k between 1 and p, $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$, that is, $r_{ik} = 1$ and $s_{kj} = 1$. In other words, if $r_{ik} = 1$ and $s_{kj} = 1$ then $t_{ij} \leftarrow 1$. This is identical to the condition needed for $M_R \odot M_S$ to have a 1 in position (i,j).

Hence we can say that:

$$M_{S \circ R} = M_R \odot M_S.$$

In the special case when we have $S = R$, $S \circ R = R^2$ and $M_{S \circ R} = M_{R^2} = M_R \odot M_R$.

§ **Theorem 2**

Let A, B, C and D be sets, R a relation from A to B, S a relation from B to C and T a relation from C to D. Then

$$T \circ (S \circ R) = (T \circ S) \circ R$$

Proof

Let the Boolean matrices for the relations R, S and T be M_R, M_S and M_T respectively. As was shown in Example 2, the Boolean matrix product represents the matrix of composition, i.e. $M_{S \circ R} = M_R \odot M_S$.

Thus we have:

$$M_{T\circ(S\circ R)} = M_{S\circ R} \odot M_T = (M_R \odot M_S) \odot M_T$$

Similarly we have:

$$M_{(T\circ S)\circ R} = M_R \odot M_{T\circ S} = M_R \odot (M_S \odot M_T)$$

Now, we know that Boolean multiplication is associative. This implies:

$$(M_R \odot M_S) \odot M_T = M_R \odot (M_S \odot M_T)$$

$$\Rightarrow M_{T\circ(S\circ R)} = M_{(T\circ S)\circ R}$$

Now since the Boolean matrices for these relations are the same,

$$\Rightarrow T \circ (S \circ R) = (T \circ S) \circ R$$

This completes the proof.

✠*Note:*

In general $R \circ S \neq S \circ R$. This can be shown by the following example.

Let $A = \{a, b\}, R = \{(a, a), (b, a), (b, b)\}$ and $S = \{(a, b), (b, a), (b, b)\}$. In this case $S \circ R = \{(a, b), (b, a), (b, b)\}$, while $R \circ S = \{(a, a), (a, b), (b, a), (b, b)\}$. Here, we can clearly see that $R \circ S \neq S \circ R$.

§ **Theorem 3**

Let A, B and C be three sets, R a relation from A to B, and S a relation from B to C. Then $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Proof

Let $c \in C$ and $a \in A$. Then $(c, a) \in (S \circ R)^{-1}$ if and only if $(a, c) \in S \circ R$. This is true if and only if there is a $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$. Finally, this is equivalent to the statement that $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$; that is, $(c, a) \in R^{-1} \circ S^{-1}$ (by the definition of composition). This proves that

$$(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$$

Conversely, let $(c, a) \in R^{-1} \circ S^{-1}$, then for some $b \in B$, $(c, b) \in S^{-1}$ and $(b, a) \in R^{-1}$ (again by the definition of composition). This implies that, $(a, b) \in R$ and $(b, c) \in S$. Hence we can say that $(a, c) \in S \circ R$, which also means that $(c, a) \in (S \circ R)^{-1}$. This proves that

$$R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}$$

Since $(S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$ and $R^{-1} \circ S^{-1} \subseteq (S \circ R)^{-1}$, we can say that:

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

This completes the proof.

2 Transitive Closure

A relation R is said to be *transitive* if for every $(a, b) \in R$ and $(b, c) \in R$ there is a $(a, c) \in R$.

A *transitive closure* of a relation R is the smallest transitive relation containing R.

Suppose that R is a relation defined on a set A and that R is not transitive. Then the transitive closure of R is the connectivity relation R^∞. We will now try to prove this claim.

§ **Theorem 4**

Let R be a relation on a set A. Then R^∞ is the transitive closure of R.

Proof

We need to prove that R^∞ is transitive and also that it is the smallest transitive relation containing R.

If a and $b \in A$, then $aR^\infty b$ if and only if there exists a path in R from a to b. If $aR^\infty b$ and $bR^\infty c$, then we can say that $aR^\infty c$. This is because $aR^\infty b$ means that there exists a path from a to b in R, similarly $bR^\infty c$ means that there exists a path from b to c in R. Hence there will also exist a path form a to c in R. (We can simply start along the path from a to b and then continue along the path from b to c. This will give us the path from a to c.) This proves that R^∞ is transitive.

Now let us consider a transitive relation S (containing R) i.e. $R \subseteq S$. Since S is transitive we can say that $S^n \subseteq S \; \forall \; n$. (This means that if there is a path of length n from a to b, then aSb, which is true as S is a transitive relation.) Now, $S^\infty = \bigcup_{n=1}^{\infty} S^n$. Hence $S^\infty \subseteq S$. Since $R \subseteq S$, therefore $R^\infty \subseteq S^\infty$, and as $S^\infty \subseteq S$, we can say that $R^\infty \subseteq S$. This means that R^∞ is the smallest of all transitive relations on A that contain R.

As R^∞ satisfies both the properties, we can say that R^∞ is the transitive closure of R on set A. This completes our proof.

✠*Note:*
If we include the identity relation Δ then $R^\infty \cup \Delta$ is the *reachability* relation R^*.

† **Example 3:**

Let $A = \{1, 2, 3, 4\}$, and let $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$. Find the transitive closure of R.

Solution:
Method 1: Using Digraph

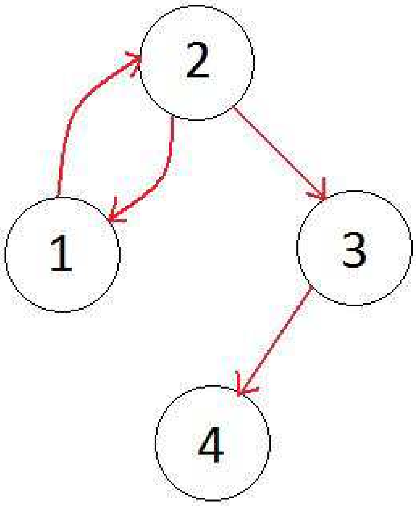

Figure 1: Digraph of R

We can determine R^∞ by geometrically computing all paths from the digraph. From vertex 1 we have paths to vertices 1, 2, 3 and 4. So the ordered pairs (1, 1), (1, 2), (1, 3) and (1,4) $\in R^\infty$. From vertex 2 we have paths to vertices to vertices 1, 2, 3 and 4. This gives us the ordered pairs (2, 1), (2, 2), (2, 3) and (2, 4). From vertex 3 we have only one path to vertex 4. This gives us the ordered pair (3, 4). From vertex 4 we do not have any paths. So we have

$$R^\infty = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,4)\}$$

Method 2: Using Matrices

Writing down the Boolean matrix we get

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If we compute the higher powers of M_R we get:

$$(M_R)^2_\odot = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^3_\odot = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)^4_\odot = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we observe that $(M_R)^n_\odot$ equals $(M_R)^2_\odot$ if n is even, and equals $(M_R)^3_\odot$ if n is odd and greater than 1. Hence we get,

$$M_{R^\infty} = M_R \vee (M_R)^2_\odot \vee (M_R)^3_\odot$$

Thus,

$$M_{R^\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is nothing but the matrix representation of the relation we obtained earlier by the digraph method (1). Thus we see that we need not consider all the powers of R^n to obtain R^∞ when the set A is finite. This result always holds good, as we will now prove.

§ **Theorem 5**

Let A be a set with $|A| = n$, and let R be a relation on A. Then

$$R^\infty = R \cup R^2 \cup \ldots \cup R^n$$

i.e. powers of R greater than n need not be considered to compute R^∞.

Proof

Let a and $b \in A$, and let $a, x_1, x_2, \ldots, x_m, b$ be a path from a to b in R; i.e. $(a, x_1), (x_1, x_2), \ldots, (x_m, b) \in R$. Now if x_i and x_j correspond to the same vertex for some $i < j$, then the path from a to b can be distinctly divided into three regions. First, a path from a to x_i; second, a path from x_i to x_j; and lastly, a path from x_j to b. Here we see that the second path forms a closed loop as $x_i = x_j$. So we can eliminate it altogether and put the first and third paths together to give us a shorter path. In a similar manner we can keep eliminating all the loops we get later on in the path too, to give us a path $a, x_1, x_2, \ldots, x_k, b$ where all of $x_1, x_2, \ldots, x_k$ are distinct. This path is the shortest one possible from a to b.

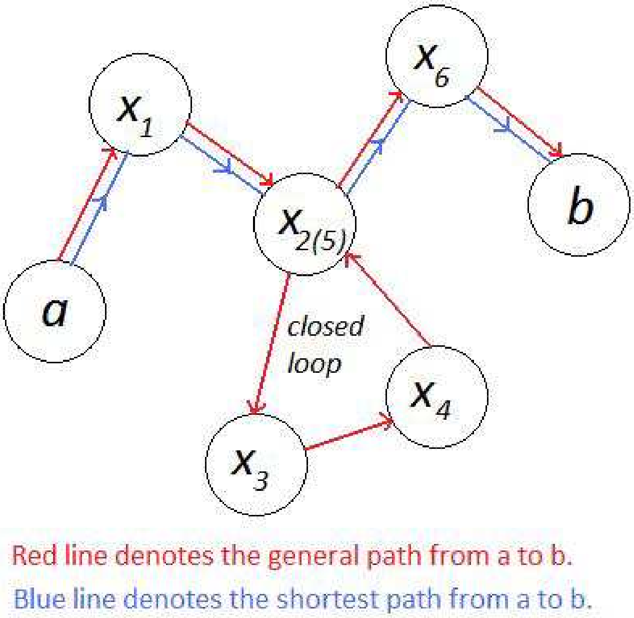

Figure 2: Shows how a closed loop can be eliminated to give the shortest path. Here, $i = 2$ and $j = 5$.

Now let us consider the case when $a \neq b$. Since the total number of elements in the set A is n, the maximum path length we can get is $n - 1$. If $a = b$, then we get the maximum path length as n (as $|A| = n$ and all the vertices except a and b are distinct). *'There is a path from a to b in R'* is equivalent to $aR^{\infty}b$. And if $aR^{\infty}b$ (i.e. there is a path from a to b in R), from the preceding discussion we know that $aR^k b$ for some $k, 1 \leq k \leq n$ (as the maximum path length possible is n). Thus $R^{\infty} = R \cup R^2 \cup \ldots \cup R^n$. Hence the theorem is proved.

3 Warshall's Algorithm

Need for Warshall's Algorithm

The methods used to solve Example 3 have certain drawbacks. The graphical method is unsystematic and impractical for large sets. The matrix method is better than the graphical method and can be implemented with the help of a program, but it tends to become costly in terms of time and space requirement in the case of large matrices. So this method is inefficient too. The Warshall's algorithm as described next helps to overcome these drawbacks.

The Procedure

Let A be a set such that $A = \{a_1, a_2, \ldots, a_n\}$ and R be a relation defined on A. Considering a general path $x_1, x_2, \ldots, x_m$ in R, all vertices except the two at the extremes, i.e. x_1 and x_m, are called the *interior vertices*. Now, we define a Boolean matrix W_k $(1 \leq k \leq n)$ as follows. W_k has a 1 in position (i, j) if and only if there is a path from a_i to a_j in R whose interior vertices, if any, come from the set $\{a_1, a_2, \ldots, a_k\}$.

If we take $k = n$, any vertex must come from the set $\{a_1, a_2, \ldots, a_n\}$, and hence W_n has a 1 in position (i, j) if and only if some path in R connects a_i with a_j. This means that $W_n = R^\infty$. If we take W_0 equal to M_R, we will get a sequence $W_0, W_1, W_2, \ldots, W_n$, where W_0 corresponds to M_R and W_n corresponds to M_{R^∞}. The *Warshall's algorithm* gives us a way to compute each matrix W_k from the previous matrix W_{k-1}. Hence we begin with the matrix representation of R and proceed to R^∞. The computation of W_k is different from that of the powers of M_R and saves time considerably as the steps involved are fewer and less complicated.

Let us suppose $W_k = [p_{ij}]$ and $W_{k-1} = [s_{ij}]$. If $p_{ij} = 1$, then there exits a path from a_i to a_j in R whose interior vertices are from $\{a_1, a_2, \ldots, a_k\}$. If the vertex a_k is not an interior vertex in this path, then all the interior vertices are from $\{a_1, a_2, \ldots, a_{k-1}\} \Rightarrow s_{ij} = 1$. If the vertex a_k is an interior vertex in the path, then there will be a subpath from a_i to a_k and another subpath from a_k to a_j. All the interior vertices in the path from a_i to a_j are distinct (Theorem 5). So a_k appears in the path only once, and hence all the interior vertices in the two subpaths mentioned earlier come from the set $\{a_1, a_2, \ldots, a_{k-1}\}$. This means that $s_{ik} = 1$ and $s_{kj} = 1$.

Thus $p_{ij} = 1$ only under two conditions. *If and only if either* $s_{ij} = 1$*, OR* $s_{ik} = 1$ *and* $s_{kj} = 1$. So if W_{k-1} has a 1 in the position (i, j), so will W_k. A new 1 can be added at position (i, j) in W_k if and only if column k in W_{k-1} has a 1 at position i and row k of W_{k-1} has a 1 at position j. So the procedure for calculating W_k from W_{k-1} is :

1. Transfer to W_k all the 1's in W_{k-1}.

2. List the locations $y_1, y_2, \ldots$, in column k of W_{k-1} where the entry is 1, and the locations $z_1, z_2, \ldots$, in row k of W_{k-1} where the entry is 1.

3. Put 1's at all positions (y_i, z_j) of W_k (if not already present).

† **Example 4:**

Find the transitive closure of R defined in Example 3.
Solution:

$$W_0 = M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we have $n = 4$.

To find W_1, $k = 1$. We can see that W_0 has 1's in column 1 at location 2, and in row 1 at location 2. Thus W_1 has a new 1 at position (2, 2).

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_2, $k = 2$. W_1 has 1's in column 2 at locations 1 and 2, and in row 2 at locations 1, 2 and 3. So the new 1's would go to positions (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (2, 3) (if not already there).

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_3, $k = 3$. W_2 has 1's in column 3 at locations 1 and 2, and in row 3 at location 4. So the new 1's would come at positions (1, 4) and (2, 4) (if not already there).

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For W_4, $k = 4$. W_3 has 1's in column 4 at locations 1, 2 and 3 but no 1's in row 4. So no new 1's are added. Hence $W_4 = W_3$. This gives us the matrix representation of R^∞ which is the same as that obtained in Example 3.

The Algorithm

To find the matrix CLOSURE of the transitive closure of a relation R whose $n \times n$ matrix representation is MAT.

WARSHALL

1. CLOSURE $\leftarrow$ MAT
2. for $k = 1$ to n
3.   for $i = 1$ to n
4.   for $j = 1$ to n
5.   CLOSURE$(i, j) \leftarrow$ CLOSURE$(i, j) \vee$(CLOSURE$(i, k) \wedge$CLOSURE(k, j))

This algorithm involves three *for* loops, with two of them nested. Each loop iterates from 1 to n. This gives us a time complexity of O(n^3). If we were to find the transitive closure using the matrix multiplication method we would get a time complexity of O(n^4). Each time a matrix multiplication is performed the time complexity is O(n^3) as there are three loops running (two nested) from 1 to n. The matrix multiplications are carried out a total of $n - 1$ times to find matrices $(M_R)^2_\odot, (M_R)^3_\odot, \ldots, (M_R)^n_\odot$, since $M_{R^\infty} = M_R \vee (M_R)^2_\odot \vee \ldots \vee (M_R)^n_\odot$. So the number of steps involved are $n^3(n-1)$, giving us a time complexity of O(n^4). Thus we see that Warshall's algorithm is surely simpler and more efficient than the matrix multiplication method.

An Application of Warshall's Algorithm
§ Theorem 6

If R and S are equivalence relations on a set A, then the smallest equivalence relation containing both R and S is $(R \cup S)^\infty$.

Proof

We know that a relation is reflexive if and only if it contains the identity or equality relation Δ. Since both R and S are reflexive, $\Delta \subseteq R$ and $\Delta \subseteq S$. This implies that $\Delta \subseteq R \cup S \subseteq (R \cup S)^\infty$. So $(R \cup S)^\infty$ is also reflexive.

R and S are symmetric. Let us consider $(a, b) \in R \Rightarrow (b, a) \in R$ (as R is symmetric). Since $R \subseteq (R \cup S) \subseteq (R \cup S)^\infty$, therefore (a, b) and $(b, a) \in (R \cup S)^\infty$. This implies that $(R \cup S)^\infty$ is also symmetric. This can be proved in a similar manner by taking $(a, b) \in S$ instead of R.

The property of transitive closure tells us that any relation T^∞ is the smallest transitive relation containing the relation T. Applying this property to $R \cup S$, we can conclude that $(R \cup S)^\infty$ is the smallest transitive relation containing $(R \cup S)$. Since it is transitive also, it is an equivalence; and since it contains $(R \cup S)$, it contains both R and S. Hence we have proved that $(R \cup S)^{\infty}$ is the smallest equivalence relation containing both R and S.

† **Example 5:**

Let $A = \{1, 2, 3, 4, 5\}, R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$, and $S = \{(1, 1), (2, 2), (3, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$ (where R and S are equivalence relations) . The partition A/R of A corresponding to R is $\{\{1, 2\}, \{3, 4\}, \{5\}\}$, and the partition A/S of A corresponding to S is $\{\{1\}, \{2\}, \{3\}, \{4, 5\}\}$. Find the smallest equivalence relation containing R and S, and compute the partition of A that it produces.

Solution:

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So,

$$M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We now compute $M_{(R \cup S)^{\infty}}$ using Warshall's algorithm. First $W_0 = M_{R \cup S}$. Next we compute W_1 ($k = 1$). W_0 has 1's at locations 1 and 2 in column 1, and at locations 1 and 2 in row 1. No new 1's need to be added as 1's are already present at positions (1, 1), (1, 2), (2, 1) and (2, 2). So $W_1 = W_0$.

We now compute W_2 ($k = 2$). Since W_1 has 1's at locations 1 and 2 of column 2, and at locations 1 and 2 of row 2, no new 1's need to be added. So $W_2 = W_1$.

Next we move on to W_3 ($k = 3$). Since W_2 has 1's at locations 3 and 4 of column 3, and at locations 3 and 4 of row 3, no new 1's need to be added. So $W_3 = W_2$.

Now we compute W_4 ($k = 4$). W_3 has 1's at locations 3, 4 and 5 of column 4, and at locations 3, 4 and 5 of row 4. So new 1's need to be added at positions (3, 5) and (5, 3) of W_3 to get W_4. Thus

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

To compute W_5 ($k = 5$), we see that, since W_4 has 1's at locations 3, 4 and 5 of column 5, and at locations 3, 4 and 5 of row 5, no new 1's need to be added. So $W_5 = W_4$.

$\therefore (R \cup S)^\infty = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$.

The corresponding partition is $\{\{1,2\},\{3,4,5\}\}$.

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