Mathematical Structures

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- Name : Ramdutt Sharma .
- Roll No.: 07CS1025
- Prof. : Mr. Niloy Ganguly
- Course : Discrete Structures (CS21001)

Mathematical Structures :

Collection of mathematical objects with operations (defined on them) & accompanying properties form a mathematical structure or system .

The mathematical objects may be sets or matrices.

These objects may be classified further (sets as finite or infinite sets , matrices as boolean or symmetric matrices etc.) .

Some example for mathematical structures are :

- collection of sets with the operations of union , intersection and complement (and there accompanying is a discrete mathematical structure) , this structure can be denoted as (sets,$\cap$,$\cup$, complement) .
- collection of 3×3 matrices with the operations of addition , multiplication and transpose .

Closure :

A structure is **closed with respect to an operation** if that operation always produces another member of that structure .

Operations :

- **Binary operation :** an operations that combines two objects .

  Ex. set intersection or multiplication .
- **Unary operations :** an operation that requires only one object .

Ex. transpose of a matrix .

Common properties of binary operations :

(Lets assume x & y be objects of structure , $^\circ$ be a unary and $\bigcirc$ & $\triangle$ be two binary operators defined on same structure ,)

Commutative : if the order of objects does not affect the outcome of a binary operation, we say that the operations is commutative .

Ex. 1. join and meet of boolean matrices are commutative operation.

$$\mathbf{A} \vee \mathbf{B} = \mathbf{B} \vee \mathbf{A}$$

$$\mathbf{A} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{A}$$

2. ordinary matrix multiplication is not commutative

$$AB \neq BA$$

Associative :

$$if (x \triangle y) \triangle z = x \triangle (y \triangle z)$$

Ex. union of sets is an associative operator .

Distributive :

$$if$$

$$x \bigcirc (y \triangle z) = (x \bigcirc y) \triangle (x \bigcirc z)$$

the we say that $\bigcirc$ is distributes over $\triangle$.

Ex. In case of real no. a,b,c;

$$a.(b + c) = a.b + a.c$$

De Morgan's rule :
Several of the structures we are going to see have a unary operator & two binary operator defined on them .

For these structure we can check whether they possess the properties of de morgan's laws .

If the unary operator is $^\circ$ & the binary operators are $\bigcirc$ & $\triangle$, then according to De morgan's law

$$(x \bigcirc y)^\circ = x^\circ \triangle y^\circ$$

$$and$$

$$(x \triangle y)^\circ = x^\circ \bigcirc y^\circ$$

Ex.sets satisfy the de morgan's laws for union , intersection and complement :

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$and$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Ex. The structure (real numbers , + , * , $\sqrt{}$) doesn't satisfy De morgan's law since $\sqrt{x+y} \neq \sqrt{x} * \sqrt{y}$.

Identity for operation : A structure with the binary operation $\triangle$ may contain a distinguished object e ,with the property x$\triangle$e = e$\triangle$x = x for all x in collection . We call e an identity for $\triangle$.

Theorem 1 :
If e is an identity for a binary operation $\triangle$, then e is unique .

Proof : Assume another object i also has the identity property , so x$\triangle$i = i$\triangle$x = x .

Then $e \triangle i = e$, but since e is an identity for $\triangle$, $i \triangle e = e \triangle i = i$. Thus $i = e$. Therefor there is at most one object with the identity property for $\triangle$.

Theorem 2 :

If $\triangle$ is an associative operation and x has a $\triangle$ inverse y , than y is unique .

Proof :

Assume there is another $\triangle$-inverse for x , say z .Then $(z \triangle x) \triangle y = e \triangle y = e$ and $z \triangle (x \triangle y) = z \triangle e = z$.
Since $\triangle$ is associative so $(z \triangle x) \triangle y = z \triangle (x \triangle y)$ and so $y=z$.