Growth Of Functions

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October 15, 2008

1 $f = O(g)$ if $\exists k$ and $c > 0$, such that $f(n) \leq cg(n) \ \ \forall n \geq k$.

Question:

$$\begin{aligned} f(n) &= \frac{1}{2}n^3 + \frac{1}{2}n^2 \\ g(n) &= n^3 \end{aligned}$$

Prove that $f = O(g)$.
Sol:

$$\begin{aligned} f(n) &= \frac{1}{2}n^3 + \frac{1}{2}n^2 \\ &\leq \frac{1}{2}n^3 + \frac{1}{2}n^3 \quad \forall n \geq 1 \end{aligned}$$

Thus $f(n) = O(g(n))$ where $c = 1, k = 1$.

2 f and g are of the same order if
$f = O(g)$ and $g = O(f)$

Question:

$$\begin{aligned} &f(n) = 3n^4 - 5n^2 \\ &g(n) = n^4 \end{aligned}$$

Prove that f and g are of the same order.
Sol:
(i)

$$\begin{aligned} f(n) &= 3n^4 - 5n^2 \\ &\leq 3n^4 \qquad \forall n \geq 1 \end{aligned}$$

Thus $f(n) = O(g(n))$ where $c = 3, k = 1$.
(ii)

$$\begin{aligned} g(n) &= n^4 \\ &= 3n^4 - 2n^4 \\ &= 3n^4 - 2n^2n^2 \\ &\leq 3n^4 - 5n^2 \quad \forall n \geq 2 \end{aligned}$$

Thus $g(n) = O(f(n))$ where $c = 1, k = 2$.
From (i) and (ii), we conclude that f and g are of same order.

3 g is of lower order than f if $\forall c$, $\exists k$ such that $f(n) \geq g(n)$ $\forall n \geq k$.
Examples:
n^4 is lower order than n^7
$nlgn$ is lower order than n^2

4 $f = \Theta(g)$ if and only if f and g are of the same order.

Theorem 0.1. *The relation Θ is an equivalence relation.*

Proof. I. Reflexive
Clearly $f = \Theta(f)$, so Θ is reflexive.
II. Symmetric

$$\begin{aligned} & f = \Theta(f) \\ \implies & f = O(g) \text{and} g = O(f) \\ \implies & g = \Theta(f) \end{aligned}$$

Thus, Θ is symmetric.
III. Transitive
Let $g = \Theta(f)$ and $h = \Theta(g)$
Thus,

$$\begin{aligned} f(n) &\geq c_1 g(n) \quad \forall n \geq k_1 \\ g(n) &\geq c_2 f(n) \quad \forall n \geq k_2 \end{aligned}$$

and

$$g(n) \geq c_3 h(n) \quad \forall n \geq k_3$$
$$h(n) \geq c_4 h(n) \quad \forall n \geq k_4$$

Therefore,

$$f(n) \geq c_3 c_1 h(n) \quad \forall n \geq max(k_1, k_3)$$
$$h(n) \geq c_2 c_4 f(n) \quad \forall n \geq max(k_2, k_4)$$

Thus,

$$h = \Theta(f)$$

Θ is thus transitive
Combining I, II, and III
Θ is an equivalence relation. □

Basic Tenets:
1.$\Theta(1)$ for constant functions
2.$\Theta(lgn) < \Theta(n^k), k > 0$
3.$\Theta(n^a) < \Theta(n^b), 0 < a < b$
4.$\Theta(a^n) < \Theta(b^n), 0 < a < b$
5.$\Theta(n^k) < \Theta(a^n), a > 1$
6. If r is not zero, $\Theta(rf) = \Theta(f)$
7. If h is a non-zero function, and $\Theta(f) = (<)\Theta(g)$, then $\Theta(fh) = (<)\Theta(gh)$
8. If $\Theta(f) < \Theta(g)$, then $\Theta(f + g) = \Theta(g)$

Question:

$$f(n) = 4n^4 - 6n^7 + 25n^3 \implies f = \Theta(n^7)$$
$$g(n) = lgn - 3n \implies g = \Theta(n)$$
$$h(n) = 1.1^n + n^{15} \implies h = \Theta(1.1^n)$$

Question:
Arrange in ascending order
1. $\Theta(nlgn)$
2. $\Theta(1000n^2 - n)$

3. $\Theta(n^{0.2})$
4. $\Theta(1,000,000)$
5. $\Theta(1.3^n)$
6. $\Theta(n + 10^7)$

Sol:
1 - D
2 - E
3 - B
4 - A
5 - F
6 - C