Discrete Structures
Lecture Dated on 08/09/08
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1 Theorem 1:-

1.1 Statement:-

Suppose R and S are relations fom A to B then:

1. If $R \subseteq S$, then $R^{-1}S^{-1}$
2. If $R \subseteq S$, then $\overline{S} \subseteq \overline{R}$
3. $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ *and* $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
4. $\overline{(R \cap S)} = \overline{R} \cup \overline{S}$ *and* $\overline{(R \cup S)} = \overline{R} \cap \overline{S}$

1.2 Proof:-

1. $(a, b) \in R \in S$

   For every $(b, a) \in R^{-1}$ it belongs to S^{-1}

$\Rightarrow (b, a) \in S^{-1}$

2. This part is a special case of general set properties
3. Suppose $(a, b) \in (R \cap S)^{-1}$ *then* (b,a) $\in (R \cap S)$, so (b,a) $\in R$ *and* $(b, a) \in S$

   This means that $(a, b) \in R^{-1}$ *and* $(a, b) \in S^{-1}$, *so* $(a, b) \in R^{-1} \cap S^{-1}$

   The converse containment can be proved by reversing the steps.

   A similar argument works to show that $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
4. This part is a special case of general set properties

2 Theorem 2:-

2.1 Statement:-

Let R and S be relations on set A then:-

1. If R is reflexive so is R^{-1}
2. If R and S are reflexive , then so are $R \cap S$ *and* $R \cup S$
3. R is reflexive if and only if $\overline{R}$ is irreflexive.

2.2 Proof:-

1. Let Δ be the equality relation on A. We know that for R to be reflexive $\Delta \subseteq R$
   But , We know that $\Delta = \Delta^{-1}$
   $So\ if \Delta \subseteq R$ then $\Delta = \Delta^{-1} \subseteq R^{-1}$ So R^{-1} is also Reflexive.

2. If R and S are reflexive then:-
   $\Delta \subseteq R$ and $\Delta \subseteq S$
   $\Rightarrow \Delta \subseteq R \cap S$ and $\Delta \subseteq R \cup S$
   Hence $R \cap S$ and $R \cup S$ are also reflexive.

3. For a Relation S to be irreflexive it has to satisfy $S \cap \Delta = \phi$
   A Relation R is reflexive if and only if $\Delta \subseteq S$
   That is if and only if $\Delta \cap \overline{R} = \phi$
   That is if and only if $\overline{R}$ is irreflexive .

3 Theorem 3:-

3.1 Statement:-

Let R be a relation on set A. Then:-

1. R is symmetric if and only if $R = R^{-1}$

2. R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta$

3. R is asymmetric if and only if $R \cap R^{-1} = \phi$

4. If R is symmetric, so are R^{-1} and $\overline{R}$

5. If R and S are symmetric, so are $R \cap S$ and $R \cup S$

3.2 Proof:-

1. For every $(a, b) \in R$
   $\Rightarrow (b, a) \in R$ because R is symmetric
   $\Rightarrow (a, b) \in R^{-1}$
   $\Rightarrow R \subseteq R^{-1}$
   Similarly, For every $(a, b) \notin R$
   $\Rightarrow (b, a) \notin R$
   $\Rightarrow (a, b) \notin R^{-1}$
   $\Rightarrow$ For every $(a, b) \in R$ $(a, b) \in R^{-1}$ and for every $(a, b) \notin R$ $(a, b) \notin R^{-1}$
   Hence we get $R = R^{-1}$ if R is symmetric
   Similarly We can prove R is symmetric if $R = R^{-1}$
   Thus we get R is symmetric if and only if $R = R^{-1}$

2. Let $(a, b) \in R$
   case 1:-
   $a \neq b$
   $\Rightarrow (b, a) \notin R$
   $\Rightarrow (a, b) \notin R^{-1}$
   For Every $(a, b) \in R$ $(a, b) \notin R^{-1}$
   $\Rightarrow$R$\cap R^{-1} = \phi$ For this case
   case 2:- $a = b$
   $\Rightarrow (a, a) \in R^{-1}$
   $\Rightarrow R \cap R^{-1} \subseteq \Delta$
   From Case 1 and Case 2 We get $R \cap R^{-1} \subseteq \Delta$ if R is antisymmetric
   Similarly We can show that R is antisymmetric if $R \cap R^{-1} \subseteq \Delta$
   Thus we get R is antisymmetric if and only if $R \cap R^{-1} \subseteq \Delta$

3. Let $(a, b) \in R$
   $\Rightarrow (b, a) \notin R$
   $\Rightarrow (a, b) \notin R^{-1}$
   For Every $(a, b) \in R$ $(a, b) \notin R^{-1}$
   $\Rightarrow R \cap R^{-1} = \phi$ if R is asymmetric
   Similarly We can show that R is symmetric if $R \cap R^{-1} = \phi$
   Thus we get R is asymmetric if and only if $R \cap R^{-1} = \phi$

4. If R is symmetric, then $R = R^{-1}$ and thus $(R^{-1})^{-1} = R = R^{-1}$ which means that R^{-1} is also symmetric.
   Also if $(a, b) \in (\overline{R})^{-1}$ if and only if $(b, a) \in \overline{R}$ if and only if $(b, a) \notin R$ if and only if $(a, b) \notin R^{-1} = R$ if and only if $(a, b) \in \overline{R}$
   $\Rightarrow$R^{-1} and $\overline{R}$ are symmetric

5. We have
   $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
   But as R and S are symmetric we have $R = R^{-1}$ and $S = S^{-1}$
   Therefore
   $(R \cap S)^{-1} = R^{-1} \cap S^{-1} = R \cap S$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1} = R \cup S$
   $\Rightarrow R \cap S$ and $R \cup S$ are also symmetric.

4 Theorem 4:-

4.1 Statement:-

Let R and S be Relations on A

1. $(R \cap S)^2 \subseteq R^2 \cap S^2$.

2. If R and S are transitive so is $R \cap S$

3. If R and S are equivalence relations , so is $R \cap S$

4.2 Proof :-

1. Geometrically we have $a(R \cap S)^2 b$ if and only if there is a path of length 2 from a to b in $R \cap S$. Both edges of this path lie in R and in S, so aR^2b and aS^2b, which implies that $a(R^2 \cap S^2)b$
   Therefore, $(R \cap S)^2 \subseteq R^2 \cap S^2$.

2. A relation T is transitive if and only if $T^2 \subseteq T$. If R and S are transitive ,then $R^2 \subseteq R$ and $S^2 \subseteq S$ so $(R \cap S)^2 \subseteq R^2 \cap S^2 \subseteq R \cap S$ so $R \cap S$ is transitive.

3. From the previous theorems we have that
   If R and S are Reflexive then $R \cap S$ is Reflexive
   If R and S are Symmetric then $R \cap S$ is Symmetric
   If R and S are Transitive then $R \cap S$ is Transitive

   Therefore We have if R and S are Equivalence relations then $R \cap S$ is also an Equivalence relation.

5 Closure:-

The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

1. $R \cup \Delta$ is the reflexive closure of R
2. $R \cup R^{-1}$ is the symmetric closure of R
3. R^{inf} is the transitive closure of R.