DISCRETE STRUCTURES

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EQUIVALENCE RELATION :

If a relation R satisfies the following :

a) Reflexivity
b) Symmetry
c) Transitivity

then R is an *equivalent relation* OR R is said to satisfy *equivalence*.

a) **Reflexivity** : A relation R on a set A is reflexive if (a,a) ∈ R for all a ∈ A, i.e. a R a for all a ∈ A.

b) **Symmetry** : A relation R on a set A is symmetric if whenever a R b, then b R a ,i.e. a R b => b R a.

c) **Transitivity** : A relation R on a set A is transitive if whenever a R b and b R c then a R c , i.e. a R b & b R c => a R c.

Example 1 : Let A be the set of all triangles in a plane.
R = { (a,b) ∈ A x A if a ≡ b}
Show that R is an equivalence relation.

\>> a) a ≡ a (a is congruent to itself) => REFLEXIVE
b) a ≡ b <=> b ≡ a ; so SYMMETRIC
c) if a ≡ b and b ≡ c then a ≡ c; so TRANSITIVE.
Hence, R is an equivalence relation.

Example 2 : Let A = Z and R be a relation on A such that a R b if a ≤ b. Is R equivalent ?

\>> a) a ≤ a => a ∈ R(a), so reflexive.
b) a ≤ b <≠> b ≤ a
=> a R b <≠> b R a. So not symmetric.
c) a ≤ b and b ≤ c => a ≤ c
Therefore, a R b and b R c => a R c. So, transitive.
Since the relation is not symmetric, it is not an equivalent relation.

Example 3 : Let A = Z and R be a relation on A such that
R = { (a,b) ∈ A x A | a & b yield the same remainder when divided by 2}
a) a R a (since a will always yield the same remainder when divided by 2)
Hence, reflexive.

b) a R b <=> b R a (when a and b yield the same remainder when divided by 2). Hence, symmetric.

c) a R b & b R c => a R c (when a,b and c yield the same remainder when divided by 2). Hence, transitive.

Hence, the relation is an equivalence relation.

Equivalence Relations and Partitions

THEOREM : Given a partition P on A. Define a relation R such that

a R b if and only if a and b are members of the same block of P.

R is an equivalence relation on A.

Proof : REFLEXIVITY : If a $\in$ A, then clearly a is in the same block as itself; so a R a.

SYMMETRY : If a R b, then a and b are in the same block, so b R a.

So,a R b <=> bRa.

TRANSITIVITY : If a R b and b R c, then a, b and c are in the same block. Hence a R c.

So, a R b and b R c => a R c.

Since R is reflexive, symmetric and transitive, it is an equivalence relation and R is called the equivalence relation determined by P.

Example 4 : Let R be an equivalence relation on set A and let a $\in$ A and b $\in$ B, then a R b if and only if R(a) = R(b).Prove.

Proof : We will have to prove it both ways as :

a) Given R(a) = R(b) then a R b.

b) Given a R b then R(a) = R(b).

a) Given R(a) =R(b) and since R is reflexive

so, b $\in$ R(a)

so, a R b. (proved)

b) Given a R b and since R is symmetric

so, b R a

(1) b $\in$ R(a)

(2) a $\in$ R(b)

Take x ∈ R(b) , from (1) , b ∈ R(a)
so, x ∈ R(a)
so, R(b) is a subset of R(a).
Take y ∈ R(a) , from (2), a ∈ R(b)
so, y ∈ R(b)
so, R(a) is a subset of R(b).
Hence R(a)=R(b). (proved)

THEOREM : Let R be an equivalence relation on Aand P b a collection of all distinct relative sets R(a) in A. Then P is a partition of A, and R is the equivalence relation determined by P.

Proof : To prove the theorem, we must prove :
a) Every element of A belongs to some relative set.
b) If R(a) and R(b) are not identical, then R(a)∩R(b) = ϕ.
Since R is reflexive, a ∈ R(a). So, (a) is true.
For proving (b), let us consider its contrapositive statement i.e.,
if R(a)∩R(b) != ϕ , then R(a)=R(b).
Let x ∈ R(a)∩R(b)
So, x ∈ R(a) and x ∈ R(b)
so, a R x and x R b
so, a R b.
=> R(a) = R(b).
Hence proved.

QUOTIENT SET

Given set A and given equivalent relation R.
Quotient equivalent relation Q(s) = A/R --> set of subsets (partitioned by R)

How many partitions?

The number of partitions of a set having 'n' elements is given by the ***Bell Number***, B(n), which can be recursively expressed as :

$$B(n+1) = B(n) + {}^{n}C_1\, B(n-1) + {}^{n}C_2\, B(n-2) + \ldots\ldots.+ {}^{n}C_n\, B(0)$$

$$\Rightarrow\ B(n+1) = {}^{n}C_n\, B(n) + {}^{n}C_{n-1}\, B(n-1) + {}^{n}C_{n-2}\, B(n-2) + \ldots\ldots.+ {}^{n}C_0\, B(0)$$

$$\Rightarrow\ B(n+1) = \sum_{k=0}^{n} {}^{n}C_k\, B(k).$$

$$B(n) = \sum_{k=0}^{n-1} {}^{n-1}C_k\, B(k).$$

GENERAL PROCEDURE TO FIND THE PARTITION

Input : Relation R and set A
Output : Partition P

Algorithm :

```
B = A
while(B!= ϕ)
   {
       Take a ∈ B
       a = P(i)
       while(1)
          {
              X = R(a)
              P(i) = P(i) U X
              If(X != new)
                     break
          }
       P = P + P(i)
       B = B – B(i)
       i = i + 1
   }
```