CS21001 Discrete Structures

Autumn 2009–10

Tutorial: Probability, Recurrence Relations, Pigeonhole principle

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1. Solve:
   (a) $f(n) = 2f(n-1) + n$ for n ≥ 1 and $f(0) = 0$.
   (b) $f(n) = 2f(\sqrt{n}) + \log n$ for $n > 2$ and $f(2) = 0$.

2. Show that of any 5 points chosen in an unit square, there are two points which will be at most $\dfrac{1}{\sqrt{2}}$ units apart.

3. The distance travelled by a particle moving in the horizontal direction in each second is equal to twice the distance it travelled in the previous one. Let s_i denote the position of the particle in the i^{th} second. Determine s_i given that $s_0 = 3$ and $s_1 = 10$.

4. A coin lands with head with a probability p. Let t_n be the probability that after n independent tosses, the number of heads are even. Derive a recursion that relates t_n to t_{n-1}, and solve this recursion to establish the formula

$$t_n = \frac{1 + (1-2p)^n}{2}$$

5. Use generating functions to solve the following recurrence relations:
   (a) $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for n ≥ 2.
   (b) $a_0 = 1$, $a_1 = 3$, $a_n = 4a_{n-1} - 4a_{n-2}$ for n ≥ 2.

6. A salesman sells at least one car everyday for 60 consecutive days, with an average of 1.5 cars per day. Show that there must be a period of consecutive days during which he sells exactly 29 cars.

7. 51 integers are chosen from the set $\{1, 2, 3, \ldots, 100\}$. Show that one of the chosen integers is a multiple of another.

8. N queens are placed in distinct squares of an $N \times N$ chessboard, with all possible placements being equally likely. A queen is free to move along horizontally or vertically, however no diagonal movement is allowed. What is the probability that no queen intersects the rest ?

9. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He is aware that, each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path and if they agree, take that one and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path ?

10. The numbers 1 to 10 are arranged in random order around a circle. Show that there are 3 consecutive numbers whose sum is at least 17.