CS21001 Discrete Structures

Autumn 2009–10

Tutorial on : Groups & Graphs

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1. Let $(S, *)$ be a semigroup and $a \in S$. The sub-semigroup generated by a is the set $\langle a \rangle = \{a * a * a \cdots a \text{ (n times) } | n > 0\}$. S is cyclic if $S = \langle a \rangle$ for some $a \in S$. Justify which of the following semigroups is/are cyclic.
   (a) $\mathbb{N}$ under integer multiplication.
   (b) $\mathbb{Z}$ under integer addition.
   (c) $\mathbb{Z}_n$ under addition modulo n.
2. Let G be a finite multiplicative group and $h = \text{ord } a$ for some $a \in G$. Show that $a^n = e$ iff $h|n$.
3. (a) Define an operation $*$ on $\mathbb{R}$ as $x * y = x + y + xy$. Prove or disprove: $(\mathbb{R}, *)$ is a group.
   (b) Prove or disprove: $(\mathbb{R} \setminus \{-1\}, *)$ is a group.
4. Let G be an Abelian group. An element $a \in G$ is called a *torsion element* of G if ord a is finite. Prove that the set of all torsion elements of G is a subgroup of G.
5. Let G be a multiplicative group and H, K subgroups of G with $H \cap K = \{e\}$. Assume that $G = HK = \{hk | h \in H, k \in K\}$. Prove that every element $a \in G$ can be written as $a = hk$ for some unique elements $h \in H$ and $k \in K$.
6. Show that the set of all complex numbers of the form $x + \mathrm{i}y$ with x, y integers and with x even is a group under addition of complex numbers.
7. Prove that an undirected graph has an even number of vertices of odd degree.
8. Show that a graph is bipartite iff it has no odd cycles.
9. Let $G = (V, E)$ be a graph with $|V| = n$. Let the maximum degree of any node be at most d and a vertex cover of G be of size at most c. Find the maximum number of edges that G can have.
10. If α and β denote the minimum vertex cover and maximum independent set respectively of an undirected connected graph G with n vertices, then $\alpha + \beta = n$. Prove or disprove the above statement.