CS21001 Discrete Structures

Autumn 2009–10

Tutorial on : Functions

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1. Consider the following C function:

```
unsigned int f (unsigned int n)
{
   if ((n == 0) || (n == 1)) return 0;
   if ((n%2) == 0) return 1 + f(n/2);
   return 1 + f(5*n+1);
}
```

   (a) What does `f(19)` return ?

   (b) What does `f(5)` return ?

   (c) What can you conclude about `f` as a function $\mathbb{N} \to \mathbb{N}$?

2. Let $f : A \to B$ and $g : B \to C$

   (a) Prove that if the function $g \circ f : A \to C$ is injective then f is injective.

   (b) Provide an example in which $g \circ f$ is injective but g is not.

   (c) Prove that if $g \circ f$ is surjective, then g is surjective.

   (d) Give an example in which $g \circ f$ is surjective but f is not.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection not equal to the identity map. Prove that there exists $n \in \mathbb{N}$ such that $n < f(n)$ and $n < f^{-1}(n)$.

4. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be given by $f_1(a) = -a^2$ and $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $f_2(a) = \sqrt{a}$. Compute $f_1 \circ f_2$. Can $f_2 \circ f_1$ be defined ?

5. (a) Show that composition of functions is associative.

   (b) Let $f_1(x) = x + 4$, $f_2(x) = x - 4$, and $f_3 = 4x$ for $x \in \mathbb{R}$. Find $f_1 \circ f_2$, $f_2 \circ f_1$, $f_1 \circ f_1$, $f_2 \circ f_2$, $f_1 \circ f_3$, $f_3 \circ f_2$, $f_3 \circ f_1$ and $f_1 \circ f_3 \circ f_2$.

6. (a) Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Then show that, f is a one-to-one correspondence between A and B, g is a one-to-one correspondence between B and A, and each is the inverse of the other.

   (b) Let $f : A \to B$ and $g : B \to C$ be invertible. Then $g \circ f$ is invertible, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

   (c) Let $A = B = \mathbb{R}$. Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$ and let $g : B \to A$ be given by

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}.$$

   Show that both f and g are bijective functions.

7. For real numbers a, b with $a < b$, we define the closed interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ and the open interval $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. Prove that the closed interval $[0, 1]$ is equinumerous with the open interval $(0, 1)$.

8. (a) Show that $(n + a)^b = \Theta(n^b)$

   (b) Prove or disprove the following statement $a^{2n} = O(a^n)$.