CS21001:Discrete Structures

Autumn semester 2009-10

Solutions to tutorial: **Relations and Digraphs**

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1. Let $A = \{1, 2, 3, 4\}$ and let R and S be the relations on A described by

$$M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Use Warshall's algorithm to compute the transitive closure of $R \cup S$.

Solution: First calculate $M_{R \cup S} = M_R \vee M_S$

This will be the initial CLOSURE matrix of Warshall's algorithm. Apply the algorithm with $N = 4$

2. Let R and S be relations on A.

(a) If R is symmetric, so are R^{-1} and $\overline{R}$.

(b) If R and S are symmetric, so are $R \cap S$ and $R \cup S$.

Proof: If R is symmetric, $R = R^{-1}$

$\therefore (R^{-1})^{-1} = R = R^{-1}$

$\Rightarrow R^{-1}$ is also symmetric.

Also, $(a, b) \in (\overline{R})^{-1} \Leftrightarrow (b, a) \in \overline{R} \Leftrightarrow (b, a) \notin R \Leftrightarrow (a, b) \notin R^{-1} = R$

$\Leftrightarrow (a, b) \in \overline{R} \Rightarrow \overline{R}$ is symmetric.

For part (b), we use the properties

$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$ and $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

Since R and S are symmetric, we can replace R^{-1} by R and S^{-1} by S

$\Rightarrow R \cap S$ and $R \cup S$ are symmetric.

3. Let R and S be relations on A.

   (a) If R is reflexive, so is R^{-1}.

   (b) If R and S are reflexive, so are $R \cap S$ and $R \cup S$.

   (c) R is reflexive if and only if $\overline{R}$ is irreflexive.

Proof: The solution is provided as Theorem 2 in Chapter 4 of Kolman.

4. Prove that the number of partitions of a set with n elements into k subsets satisfies the recurrence relation

$$S(n, k) = S(n-1, k-1) + k.S(n-1, k)$$

Solution: There are two cases:

(a) The n^{th} element is the only element in its partition. In this case, the number of partitions is $S(n-1, k-1)$

(b) The n^{th} element is in a subset with more than one elements. In this case, the remaining elements can be in $S(n-1, k)$ partitions. The n^{th} element can be in any of these k sets.

$\therefore S(n, k) = S(n-1, k-1) + k.S(n-1, k)$

5. Let $P_1 = \{A_1, A_2, ..., A_k\}$ be a partition of A and $P_2 = \{B_1, B_2, ..., B_m\}$ a partition of B. Prove that

$$P = \{A_i \times B_j, 1 \leq i \leq k, 1 \leq j \leq m\}$$

is a partition of $A \times B$.

Solution: We have to prove that for all $1 \leq i \leq k$, $1 \leq j \leq m$, $1 \leq r \leq k$, $1 \leq s \leq m$

$$(A_i \times B_j) \cap (A_r \times B_s) = \phi$$

where $i \neq r$ and $j \neq s$

Since $A_i \cap A_r = \phi$ and $B_j \cap B_s = \phi$, there can be no two $(A_i \times B_j)$'s having

the same values i. e.

$(A_i \times B_j) = (A_r \times B_s) \Rightarrow i = r$ and $j = s$

We also have to prove that $\bigcup(A_i \times B_j) = A \times B$

To do this, we just have to prove that the number of elements on both sides

is the same which is a counting problem.

6. Prove by induction that if a relation R on a set A is symmetric, then R^n is symmetric for $n \geq 1$.

Solution: Let a and b be two elements belonging to A. Since R is symmetric, $aRb = bRa$

We have to prove that $aR^nb = bR^na$

Basis: For $n = 1$, the expression is true.

Induction step: Let us assume that the expression is true for $n = k - 1$.

$aR^kb = aR^{k-1}a_1Rb$

$\therefore aR^kb = aR^{k-1}bRa_1$

$\therefore aR^kb = bR^{k-1}aRa_1$

$\therefore aR^kb = bR^{k-1}a_1Ra$

$\therefore aR^kb = bR^ka$

Hence, proved.

7. Let $A = \{a, b, c, d, e\}$ and $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, give the relation R defined on A and its digraph.

Solution: The relation R is defined by

$$\begin{aligned} m_{ij} &= 1 \text{ if } (a_i, b_j) \in R \\ &= 0 \text{ if } (a_i, b_j) \notin R \end{aligned}$$

The digraph will have 5 vertices $V = a, b, c, d, e$ and $E = \{(i, j)$ where $m_{ij} = 1\}$

8. Let R be a relation from A to B. Prove that for all subsets A_1 and A_2 of A

$$R(A_1 \cap A_2) = R(A_1) \cap R(A_2) \qquad \text{if and only if}$$

$$R(a) \cap R(b) = \{\} \qquad \text{for any distinct } a, b \text{ in } A$$

Solution: We will break the proof down into two parts.

Let $$R(A_1 \cap A_2) = R(A_1) \cap R(A_2) \ldots (1)$$

and $$R(a) \cap R(b) = \{\} \qquad \ldots (2)$$

Necessary part: $(1) \Rightarrow (2)$

We prove this by proving the contrapositive i.e. $\sim (2) \Rightarrow \sim (1)$

$\Rightarrow$ There exist atleast two distinct elements a, b such that their image sets have elements in common.

$\Rightarrow$ If $a \in A_1$ and $b \in A_2$ then $A_1 \cap A_2$ can be equal to ϕ but the R.H.S of (1) will not be empty.

Hence, proved.

Sufficient part: $(2) \Rightarrow (1)$

$R(A_1) \cap R(A_2)$ will consist of images of only those elements which have a duplicate in the other subset. ($\because$ of (2))

$R(A_1 \cap A_2)$ will consist of images of common elements of A_1 and A_2

Hence, proved