CS21001 Discrete Structures

Autumn 2009–10

Solutions to Tutorial on : Probability, Recurrence Relations, Pigeonhole principle

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1. Solve:

   (a) $f(n) = 2f(n-1) + n$ for n ≥ 1 and $f(0) = 0$.

   Solution: This recurrence is of the form $f(n) = g(n)f(n-1) + h(n)$, $n \geq 1$, where $g(n) = 2$ and $h(n) = n$. Define, $f'(n)$ such that $f(n) = g(n)g(n-1)\ldots g(1)f'(n)$ and $f(n-1) = g(n-1)g(n-2)\ldots g(1)f'(n-1)$. Substitute $f(n)$ and $f(n-1)$ in the original recurrence, to get

   $$\prod_{i=1}^{n} g(i)f'(n) = \prod_{i=1}^{n} f'(n-1) + h(n)$$
   $$f'(n) = f'(n-1) + \frac{h(n)}{\prod_{i=1}^{n} g(i)}$$

   for $n \geq 1$. Comparing with this form, we have $f(n) = 2^n f'(n)$. Thus,

   $$\begin{aligned} f'(n) &= f'(n-1) + \frac{n}{2^n} \\ &= f'(0) + \sum_{i=1}^{n} \frac{i}{2^i} \end{aligned}$$

   Since $f'(0) = 0$, we have

   $$\begin{aligned} f(n) &= 2^n f'(n) \\ &= 2^n \sum_{i=1}^{n} \frac{i}{2^i} \\ &= 2^{n-1} - n - 2 \end{aligned}$$

   (b) $f(n) = 2f(\sqrt{n}) + \log n$ for $n > 2$ and $f(2) = 0$.

   Solution: Substitute $m = \log_2 n$. Therefore, $f(2^m) = 2f\left(2^{\frac{m}{2}}\right) + \log n$. We can now rename $g(m) = f(2^m)$. Thus the new recurrence becomes, $g(m) = 2g\left(\frac{m}{2}\right) + cm$, for $m > 1$, $g(1) = 0$ and where c is a constant. Solving the recurrence, we have

   $$\begin{aligned} g(m) &= 2g(\frac{m}{2}) + cm \\ &= 2^2 g(\frac{m}{2^2}) + c2m \end{aligned}$$

   For simplicity, let us assume for some k, $m = 2^k$. Then we have

   $$\begin{aligned} g(m) &= 2^k g(1) + ckm \\ &\leq m \log m \end{aligned}$$

   So, $g(m) = O(m \log m)$. Finally, substituting original values, we have $f(n) = O(\log n \log\log n)$.

2. Show that of any 5 points chosen in an unit square, there are two points which will be at most $\frac{1}{\sqrt{2}}$ units apart.

   Solution: Place the square with its vertices at $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$. The unit square can be covered by 4 closed discs, each of diameter $\frac{\sqrt{2}}{2}$, with centers at $\left(\pm\frac{1}{4}, \pm\frac{1}{4}\right)$. When 5 points are placed on the square, some two of them must lie in the same disc, and these points will be at most $\frac{\sqrt{2}}{2}$ .i.e. $\frac{1}{\sqrt{2}}$ units apart.

3. The distance travelled by a particle moving in the horizontal direction in each second is equal to twice the distance it travelled in the previous one. Let s_i denote the position of the particle in the i^{th} second. Determine s_i given that $s_0 = 3$ and $s_1 = 10$.

   Solution: The distance travelled by the particle in the i^{th} second is $s_i - s_{i-1}$ and the distance travelled in the $(i-1)^{th}$ second is $s_{i-1} - s_{i-2}$. By the problem, we have

$$s_i - s_{i-1} = 2\left(s_{i-1} - s_{i-2}\right)$$
$$s_i - 3s_{i-1} - 2s_{i-2} = 0$$

   The characteristic equation is $x^2 - 3x + 2 = 0$ with distinct roots $x = 1, 2$. The general solution is $s_i = c_1 + c_2 2^n$. Now, on applying the initial conditions we have

$$s_0 = 3 = c_1 + c_2$$
$$s_1 = 10 = c_1 + 2c_2$$

   Solving for c_1 and c_2, we obtain $c_1 = -4$, $c_2 = 7$. Therefore, the distance travelled by a particle in the i^{th} second, $s_i = 7.\,2^i - 4$.

4. A coin lands with head with a probability p. Let t_n be the probability that after n independent tosses, the number of heads are even. Derive a recursion that relates t_n to t_{n-1}, and solve this recursion to establish the formula

$$t_n = \frac{1 + (1-2p)^n}{2}$$

   Solution: Let X be the event that the first $(n-1)$ tosses produce an even number of heads, and let Y be the event that the n^{th} toss is a head. We can obtain an even number of heads in n tosses in two distinct ways. First, there is an even number of heads in the first $(n-1)$ tosses, and the n^{th} toss results in tails. This is the event $X \cap Y^c$. Second, there is an odd number of heads in the first $(n-1)$ tosses, and the n^{th} toss results in heads. This is the event $X^c \cap Y$. Since X and Y are independent,

$$\begin{aligned} t_n &= \Pr\left((X \cap Y^c) \cup (X^c \cap Y)\right) \\ &= \Pr\left(X \cap Y^c\right) + \Pr\left(X^c \cap Y\right) \\ &= \Pr(X)\Pr(Y^c) + \Pr(X^c)\Pr(Y) \\ &= t_{n-1}(1-p) + (1-t_{n-1})p. \end{aligned}$$

   We now use induction. For $n = 0$, we have $t_0 = 1$, which agrees with given formula for t_n. Assume, that the formula holds with n replaced by $n-1$, i.e.,

$$t_{n-1} = \frac{1 + (1-2p)^{n-1}}{2}$$

   Using this equation, we have

$$\begin{aligned} t_n &= t_{n-1}(1-p) + (1-t_{n-1})p \\ &= p + (1-2p)t_{n-1} \\ &= p + (1-2p)\frac{1 + (1-2p)^{n-1}}{2} \\ &= \frac{1 + (1-2p)^n}{2} \end{aligned}$$

   Hence, the given formula applies for all n.

5. Use generating functions to solve the following recurrence relations:

   (a) $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ for n ≥ 2.

   Solution: Let $F(x) = F_0 + F_1x + F_2x^2 + \cdots + F_nx^n + \cdots$. Then, $F(x) = F_0 + F_1x + (F_0 + F_1)x^2 + (F_1 + F_2)x^3 + \cdots + (F_{n-2} + F_{n-1})x^n + \cdots = F_0 + F_1x + x(F(x) - F_0) + x^2F(x)$, i.e., $(1 - x - x^2)F(x) = F_0 + F_1x - F_0x = x$, i.e., $F(x) = \dfrac{x}{(1-x-x^2)} = \dfrac{1}{\gamma - \gamma'}\left(\dfrac{1}{1-\gamma x} - \dfrac{1}{1-\gamma' x}\right)$, where $\gamma = \dfrac{1+\sqrt{5}}{2}$ and $\gamma' = \dfrac{1-\sqrt{5}}{2}$. We have $\gamma - \gamma' = \sqrt{5}$, and so $F(x) = \dfrac{1}{\sqrt{5}}\left[\left(1 + \gamma x + \gamma^2x^2 + \cdots + \gamma^nx^n + \cdots\right) - \left(1 + \gamma' x + \gamma'^2x^2 + \cdots + \gamma'^nx^n + \cdots\right)\right]$, i.e. $F_n = \dfrac{1}{\sqrt{5}}\left(\gamma^n - \gamma'^n\right), \forall n \in \mathbb{N}$.

   (b) $a_0 = 1, a_1 = 3, a_n = 4a_{n-1} - 4a_{n-2}$ for n ≥ 2.

   Solution: Let $a(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$. Then, $a(x) = a_0 + a_1x + (4a_1 - 4a_0)x^2 + (4a_2 - 4a_1)x^3 + \cdots + (4a_{n-1} - 4a_{n-2})x^n + \cdots = a_0 + a_1x + 4x(a(x) - a_0) - 4x^2a(x)$, i.e., $(1 - 4x + 4x^2)a(x) = a_0 + a_1x - 4a_0x = 1 - x$, i.e. $a(x) = \dfrac{1-x}{1-4x+4x^2} = \dfrac{1-x}{(1-2x)^2} = \dfrac{1}{2}\left[\dfrac{1+(1-2x)}{(1-2x)^2}\right] = \dfrac{1}{2}\left[\dfrac{1}{(1-2x)^2} + \dfrac{1}{(1-2x)}\right] = \dfrac{1}{2}\left[\left(1 + 2(2x) + 3(2x)^2 + \cdots + (n+1)(2x)^n + \cdots\right) + \left(1 + 2x + (2x)^2 + \cdots + (2x)^n + \cdots\right)\right]$, i.e. $a_n = \dfrac{1}{2}(n+1+1)2^n = (n+2)2^{n-1}, \forall n \in \mathbb{N}$.

6. A salesman sells at least one car everyday for 60 consecutive days, with an average of 1.5 cars per day. Show that there must be a period of consecutive days during which he sells exactly 29 cars.

   Solution: Let c_i be the number of cars the salesman has sold till the end of the i^{th} day. Since, the salesman sells at least one car each day and at most $60 \times 1.5 = 90$ cars in 60 days, we have

   $$1 \leq c_1 < c_2 < \ldots < c_{60} \leq 90$$

   Now, as each c_i is different, we have

   $$c_1 + 29 < c_2 + 29 < \ldots < c_{60} + 29 \leq 90 + 29 = 119$$

   There are 120 numbers, each between 1 and 119. By the pigeonhole principle, two of the numbers have to be the same. Since, $c_1 < c_2 < \ldots < c_{60}$ are all different and $c_1 + 29 < c_2 + 29 < \ldots < c_{60} + 29$ are all different, there exists c_i and c_j such that $c_i = c_j + 29$. Therefore, in between the i^{th} day and the j^{th} day, the salesman sells exactly 29 cars.

7. 51 integers are chosen from the set $\{1, 2, 3, \ldots, 100\}$. Show that one of the chosen integers is a multiple of another.

   Solution: Let the chosen integers be $a_1, a_2, \ldots, a_{51}$. Now, we can express any integer as $a_k = 2^{m_k}b_k$, where b_k is an odd integer and $k = 1, 2, \ldots, 51$. The 51 numbers $b_1, b_2, \ldots, b_{51}$ are all odd, however only 50 odd integers can exist within $\{1, 2, 3, \ldots, 100\}$. By the pigeonhole principle, some pair of the numbers b_k must be equal. Since, the chosen 51 integers are distinct and $b_i = b_j$, then $m_i \neq m_j$ which implies either a_i is a multiple of a_j or vice-versa.

8. N queens are placed in distinct squares of an $N \times N$ chessboard, with all possible placements being equally likely. A queen is free to move along horizontally or vertically, however no diagonal movement is allowed.

What is the probability that no queen intersects the rest ?

Solution: By the problem, there should be no row or column with more than one queen. The first queen can occupy any position in the $N \times N$ chessboard. Placing this queen, however eliminates one row and one column. For the second queen, we can imagine that the conflicting row and column have been removed, thus leaving us with a $(N-1) \times (N-1)$ chessboard. Therefore, the second queen has $(N-1)^2$ choices. Similarly, for the third queen we have $(N-2)^2$ choices, and so on. In the absence of any constraints, there are $N^2 \cdot (N^2-1) \cdots (N^2-(N-1)) = \dfrac{N^2!}{(N^2-N)!}$ ways in which we can place N queens. So the probability that no queen intersects each other is

$$\frac{N^2 \cdot (N-1)^2 \cdots 2^2}{\dfrac{N^2!}{(N^2-N)!}} = \frac{(N!)^2}{\dfrac{N^2!}{(N^2-N)!}}$$

9. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He is aware that, each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path and if they agree, take that one and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path ?

   Solution: Consider the sample space for the hunter's strategy. The events that lead to the correct path are:

   (a) Both dogs agree on the correct path [probability p^2, by independence].

   (b) The dogs disagree, dog A chooses the correct path, and the hunter follows dog A [probability $\frac{1}{2}p(p-1)$].

   (c) The dogs disagree, dog B chooses the correct path, and the hunter follows dog B [probability $\frac{1}{2}p(p-1)$]

   Since the above events are disjoint, so the probability that the hunter chooses the correct path is

   $$p^2 + \frac{1}{2}p(p-1) + \frac{1}{2}p(p-1) = p$$

   On the other hand, if the hunter lets one dog choose the path, this dog will also choose the correct path with probability p. Thus, the two strategies are equally effective.

10. The numbers 1 to 10 are arranged in random order around a circle. Show that there are 3 consecutive numbers whose sum is at least 17.

    Solution: Let the numbers occur around the circle as $a_1, a_2, a_3, \ldots, a_{10}$. Then the triples of the consecutive numbers are :

    $$\{a_1, a_2, a_3\}, \{a_2, a_3, a_4\}, \{a_3, a_4, a_5\} \ldots, \{a_9, a_{10}, a_1\}, \{a_{10}, a_1, a_2\}$$

    Thus we see that each number occurs in exactly 3 triples. Hence the sum of all the triples

    $$\begin{aligned} &\{a_1 + a_2 + a_3\} + \{a_2 + a_3 + a_4\} + \{a_3 + a_4 + a_5\} + \cdots + \{a_9 + a_{10} + a_1\} + \{a_{10} + a_1 + a_2\} \\ &= 3(1 + 2 + 3 + \cdots + 10) \\ &= \frac{3 \times 10 \times 11}{2} \\ &= 165 \end{aligned}$$

    Now, there are 10 triples whose sum is $165 > 10 \times 16$. Hence, there must be at one triple whose sum is at least $16 + 1 = 17$.