

CS21001:Discrete Structures

Autumn semester 2009-10

Tutorial: Methods of proof, permutations and combinations

1. Prove

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \dots + \binom{n}{r}\binom{m}{0}$$

Suppose you have $n + m$ balls. The L.H.S denotes the total number of ways in which we can choose r balls from $n + m$ balls. Now suppose we divide these $n + m$ balls into two groups of n and m balls. The number of ways of selecting a total of r balls from these two groups is: 0 from the first group and r from the second, 1 from the first groups and $r - 1$ from the second and so on. This sum is denoted by the R.H.S.

Therefore, L.H.S = R.H.S.

2. Let F_n , $n \in \mathbb{N}$, denote the sequence of Fibonacci numbers. Prove by induction on n that

$$F_{2n} = F_n(F_{n+1} + F_{n-1}) \text{ and } F_{2n+1} = F_{n+1}^2 + F_n^2 \text{ for all } n \geq 1.$$

Basis: For $n = 1$, $F_{2n} = F_2 = 1$. Also, $F_2 = F_1(F_2 + F_0) = 1$. Moreover, $F_{2n+1} = F_3 = 2 = 1^2 + 1^2 = F_{n+1}^2 + F_n^2$.

Induction: Suppose that $F_{2n} = F_n(F_{n+1} + F_{n-1})$ and $F_{2n+1} = F_{n+1}^2 + F_n^2$ for some $n \geq 1$. We have $F_{2(n+1)} = F_{2n+2} = F_{2n+1} + F_{2n} = (F_{n+1}^2 + F_n^2) + F_n(F_{n+1} + F_{n-1}) = F_{n+1}(F_n + F_{n+1}) + F_n(F_n + F_{n-1}) = F_{n+1}F_{n+2} + F_nF_{n+1} = F_{n+1}(F_{n+2} + F_n)$.

Moreover, $F_{2(n+1)+1} = F_{2n+3} = F_{2n+2} + F_{2n+1} = F_{n+1}(F_{n+2} + F_n) + (F_{n+1}^2 + F_n^2) = F_{n+1}(F_{n+1} + 2F_n) + (F_{n+1}^2 + F_n^2) = (F_{n+1}^2 + 2F_{n+1}F_n + F_n^2) + F_{n+1}^2 = (F_{n+1} + F_n)^2 + F_{n+1}^2 = F_{n+2}^2 + F_{n+1}^2$

3. Show that $2^n > n^3$ for $n \geq 10$.

Basis: For $n = 10$, $1024 > 1000$

Induction: Assume that the inequality holds for all $n \leq k$.

Now consider $(k+1)^3 = k^3 \cdot (1 + \frac{1}{k})^3$

$$< 2^k \cdot (1 + \frac{1}{k})^3$$

$$< 2^k \cdot 2 \quad (\because k \geq 10 \Rightarrow (1 + \frac{1}{k}) \leq 1.1 \Rightarrow (1 + \frac{1}{k})^3 \leq 1.331 < 2)$$

$\therefore 2^{k+1} > (k+1)^3$.

4. (a) Show that the total number of permutations of p red balls and 0, or 1, or 2 ..., or q white balls is

$$\frac{p!}{p!} + \frac{(p+1)!}{p!1!} + \frac{(p+2)!}{p!2!} + \dots + \frac{(p+q)!}{p!q!}$$

The total number of permutations of $(p+q)$ objects where p are of one kind and q are of other are $\frac{(p+q)!}{p!q!}$. The given expression is just

$$\sum_{i=0}^q \frac{(p+i)!}{p!i!}$$

- (b) Show that the sum in part (a) is

$$\frac{(p+q+1)!}{(p+1)!q!}$$

Add and subtract $\frac{(p+q+1)!}{(p+1)!q!}$ to the expression derived in part (a). Use the expression

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$$

where $n = p + q$ and $r = p + 1$. All the other terms will cancel out.

5. Prove or disprove

- (a) Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

Necessary part:

$5n + 6$ is odd.

Let $5n + 6 = 2k + 1$

$\Rightarrow 5n = 2k - 5$ which is an odd number.

$\Rightarrow n$ is an odd number.

Sufficient part:

n is odd.

Let $n = 2k + 1$

$\Rightarrow 5n + 6 = 10k + 11$

$\Rightarrow 5n + 6 = \text{even number} + \text{odd number}$

$\Rightarrow 5n + 6$ is odd.

(b) $x - y$ divides $x^n - y^n$ for $n \geq 1$.

Basis:

For $n = 1$, $x - y$ is divisible by $x - y$.

Induction:

Assume the given hypothesis is true for $n = k$ i.e. $x^k - y^k$ is divisible by $x - y$.

Consider $(x^k - y^k)(x + y) = x^{k+1} - y^{k+1} - xy(x^{k-1} - y^{k-1})$

Using strong induction, $T(k)$ and $T(k - 1)$ are true. Therefore, $T(k + 1)$ is true.

Hence, proved.

6. Prove the triangle inequality, which states that if x and y are real numbers then

$$|x| + |y| \geq |x + y|$$

(where $|x|$ represents the absolute value of x)

There are four cases to be considered:

Case 1: $x \geq 0$ and $y \geq 0$

$$|x| = x, |y| = y \text{ and } |x + y| = x + y$$

$$\therefore L.H.S = R.H.S$$

Case 2: $x < 0$ and $y < 0$

$$|x| = -x, |y| = -y \text{ and } |x + y| = -(x + y)$$

$$\therefore L.H.S = R.H.S$$

Case 3: $x \geq 0$ and $y < 0$

$$|x| = x, |y| = -y$$

There are two cases for $|x + y|$:

If $|x| \geq |y|$, then $|x + y| = x + y$

If $|x| < |y|$, then $|x + y| = -(x + y)$

In both cases, $L.H.S > R.H.S$

Case 4: $x < 0$ and $y \geq 0$

This is the same as Case 3 with the signs interchanged.

7. Five fair coins are tossed and the results are recorded.

(a) How many different sequences of heads and tails are possible?

There are two different possibilities for each coin.

\therefore Total number of sequences = $2^5 = 32$.

(b) How many of the sequences in part (a) have exactly one head recorded?

There are four tails and one head. Total number of permutations is $\frac{5!}{4!1!} = 5$.

(c) How many of the sequences in part (a) have exactly three heads recorded?

There are two tails and three heads. Total number of permutations is $\frac{5!}{3!2!} = 10$.

8. How many zeroes are there at the end of $12!?$ at the end of $26!?$ at the end of $53!?$

We just have to count the number of fives since there will always be more twos than fives. There are two fives (in 5 and 10) until 12. Therefore, $12!$ has two zeroes. $26!$ has 6 zeroes since 25 contributes two fives. $53!$ has 12 zeroes.

9. Prove by mathematical induction that if A_1, A_2, \dots, A_n and B are any $n + 1$ sets, then

$$\left(\bigcup_{i=1}^n A_i\right) \cap B = \bigcup_{i=1}^n (A_i \cap B)$$

Basis:

For $n = 1$, $L.H.S = A_1 \cap B = R.H.S$

Induction:

Suppose the hypothesis is true for $n = k$.

$$\begin{aligned} & (\bigcup_{i=1}^{k+1} A_i) \cap B \\ &= ((\bigcup_{i=1}^k A_i) \cap B) \cup (A_{k+1} \cap B) \\ &= \bigcup_{i=1}^k (A_i \cap B) \cup (A_{k+1} \cap B) \\ &= \bigcup_{i=1}^{k+1} (A_i \cap B) \end{aligned}$$

Hence, proved.