CS21001 Discrete Structures

Autumn 2009–10

Solutions to Tutorial on : Functions

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1. Consider the following C function:

```
unsigned int f (unsigned int n)
{
   if ((n == 0) || (n == 1)) return 0;
   if ((n%2) == 0) return 1 + f(n/2);
   return 1 + f(5*n+1);
}
```

   (a) What does `f(19)` return ?

   Solution: We have `f(19)` = `1 + f(96)` = `2 + f(48)` = `3 + f(24)` = `4 + f(12)` = `5 + f(6)` = `6 + f(3)` = `7 + f(16)` = `8 + f(8)` = `9 + f(4)` = `10 + f(2)` = `11 + f(1)` = 11 + 0 = 11.

   (b) What does `f(5)` return ?

   Solution: We have `f(5)` = `1 + f(26)` = `2 + f(13)` = `3 + f(66)` = `4 + f(33)` = `5 + f(166)` = $\cdots$ = `12 + f(13)` = $\cdots$ = `22 + f(13)` = $\cdots$ `32 + f(13)` = $\cdots$. Thus the above function does not terminate when 5 is passed as its argument. When the recursion stack runs out of memory, it exits with an error message (typically `segmentation fault`).

   (c) What can you conclude about `f` as a function $\mathbb{N} \to \mathbb{N}$?

   Solution: The sequence of computation in Part (b) implies that `f(13) = 10 + f(13)` i.e., `f` is not well defined as a function $\mathbb{N} \to \mathbb{N}$.

2. Let $f : A \to B$ and $g : B \to C$

   (a) Prove that if the function $g \circ f : A \to C$ is injective then f is injective.
   Solution: Try yourself !

   (b) Provide an example in which $g \circ f$ is injective but g is not.
   Solution: Take $f(x) = \sqrt{x}$, $f(x) : \mathbb{R}^+ \to \mathbb{R}^+$ and $g(x) = x^2$, $g(x) : \mathbb{R} \to \mathbb{R}$.

   (c) Prove that if $g \circ f$ is surjective, then g is surjective.
   Solution: Try yourself !

   (d) Give an example in which $g \circ f$ is surjective but f is not.
   Solution: Take $f(x) = \sqrt{x}$, $f(x) : \mathbb{R}^+ \to \mathbb{R}$ and $g(x) = x^2$, $g(x) : \mathbb{R} \to \mathbb{R}$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection not equal to the identity map. Prove that there exists $n \in \mathbb{N}$ such that $n < f(n)$ and $n < f^{-1}(n)$.

   Solution: Let $S = \{a \in \mathbb{N} | f(a) \neq a\}$. Since f is not the identity map, we have $S \neq \phi$. Let n be the minimum element in S. Thus, $f(0) = 0$, $f(1) = 1$, ... , $f(n-1) = n-1$. Since f is injective, $f(n)$ cannot be equal to $0, 1, 2, \ldots, n-1$. Moreover, since $f(n) \neq n$, we must have $f(n) > n$. Further, $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, ... , $f^{-1}(n-1) = n-1$, whereas $f^{-1}(n) = n$ (since $f(n) > n$ and f is injective). Therefore it follows that $f^{-1}(n) > n$, too.

4. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be given by $f_1(a) = -a^2$ and $f_2 : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $f_2(a) = \sqrt{a}$. Compute $f_1 \circ f_2$. Can $f_2 \circ f_1$ be defined ?

   Solution: $(f_1 \circ f_2)(a) = f_1(f_2(a)) = f_1(\sqrt{a}) = -a \ \forall a \in \mathbb{R}^+$. It is possible to define the function $f_1 \circ f_2 : \mathbb{R}^+ \to \mathbb{R}$ because the range of f_2 is $\mathbb{R}^+$, which is a subset of $\mathbb{R}$ and $\mathbb{R}$ is the domain of f_1. But, the range of f_1 is not included in the domain of f_2. So, $f_2 \circ f_1$ cannot be defined.

5. (a) Show that composition of functions is associative.

   Solution: Consider three functions $f : A \to B$, $g : B \to C$, and $h : C \to D$. Then we require to show that

$$h \circ (g \circ f) = (h \circ g) \circ f$$

   Now, let us assume that $b = f(a)$, $c = g(b)$, and $d = h(c)$. We have $(a, b) \in f$, $(b, c) \in g$, $(c, d) \in h$ and $(a, c) \in g \circ f$, $(b, d) \in h \circ g$. By the same argument we can write, $(a, d) \in h \circ (g \circ f)$. Similarly, $(a, d) \in (h \circ g) \circ f$. This being true for any a and corresponding d, proves the associativity.

   (b) Let $f_1(x) = x + 4$, $f_2(x) = x - 4$, and $f_3 = 4x$ for $x \in \mathbb{R}$. Find $f_1 \circ f_2$, $f_2 \circ f_1$, $f_1 \circ f_1$, $f_2 \circ f_2$, $f_1 \circ f_3$, $f_3 \circ f_2$, $f_3 \circ f_1$ and $f_1 \circ f_3 \circ f_2$.
   Solution:

$$\begin{aligned} f_1 \circ f_2 &= \{(x, x) | x \in \mathbb{R}\} \\ f_2 \circ f_1 &= \{(x, x) | x \in \mathbb{R}\} \\ f_1 \circ f_1 &= \{(x, x + 8) | x \in \mathbb{R}\} \\ f_2 \circ f_2 &= \{(x, x - 8) | x \in \mathbb{R}\} \\ f_1 \circ f_3 &= \{(x, 4x + 4) | x \in \mathbb{R}\} \\ f_3 \circ f_2 &= \{(x, 4x - 16) | x \in \mathbb{R}\} \\ f_3 \circ f_1 &= \{(x, 4x + 16) | x \in \mathbb{R}\} \\ (f_1 \circ f_3) \circ f_2 &= \{(x, 4x - 12) | x \in \mathbb{R}\} \end{aligned}$$

6. (a) Let $f : A \to B$ and $g : B \to A$ be functions such that $g \circ f = 1_A$ and $f \circ g = 1_B$. Then show that, f is a one-to-one correspondence between A and B, g is a one-to-one correspondence between B and A, and each is the inverse of the other.

   Solution: The assumption means that

$$g(f(a)) = a \text{ and } f(g(b)) = b, \forall a \in A, b \in B.$$

   This shows that $Ran(f) = B$ and $Ran(g) = A$, so each function is onto. If $f(a_1) = f(a_2)$, then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$. Thus f is injective. In a similar way, we see that g is injective, so both f and g are invertible. Note f^{-1} is everywhere defined since $Dom(f^{-1}) = Ran(f) = B$. Now, if b is any element in B,

$$f^{-1}(b) = f^{-1}(f(g(b))) = (f^{-1} \circ f)g(b) = 1_A(g(b)) = g(b).$$

   Thus $g = f^{-1}$, so also $f = (f^{-1})^{-1} = g^{-1}$. Then, since g and f are onto, f^{-1} and g^{-1} are onto, so f and g must be everywhere defined.

   (b) Let $f : A \to B$ and $g : B \to C$ be invertible. Then $g \circ f$ is invertible, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

   Solution: We know that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, since this is true for any two relations. Since g^{-1} and f^{-1} are functions by assumption, so is their composition, and then $(g \circ f)^{-1}$ is a function. So $g \circ f$ is invertible.

(c) Let $A = B = \mathbb{R}$. Let $f : A \to B$ be given by the formula $f(x) = 2x^3 - 1$ and let $g : B \to A$ be given by

$$g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}.$$

Show that both f and g are bijective functions.

Solution: Let $x \in A$ and $y = f(x) = 2x^3 - 1$. Then $\frac{1}{2}(y+1) = x^3$; therefore,

$$x = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}} = g(y) = g(f(x)) = (g \circ f)(x).$$

Thus $g \circ f = 1_A$. Similarly, $f \circ g = 1_B$, so by the previous proof both f and g are bijections.

7. For real numbers a, b with $a < b$, we define the closed interval $[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ and the open interval $(a, b) = \{x \in \mathbb{R} | a < x < b\}$. Prove that the closed interval $[0, 1]$ is equinumerous with the open interval $(0, 1)$.

   Solution: The inclusion map $f : (0, 1) \to [0, 1]$ taking $x \mapsto x$ is injective. Also the map $g : [0, 1] \to (0, 1)$ taking $x \mapsto \frac{1}{4} + \frac{x}{2}$ is an injective embedding of $[0, 1]$ in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ which is a subset of $(0, 1)$.

8. (a) Show that $(n + a)^b = \Theta(n^b)$

   Solution: Here we will want to find constants $c_1, c_2, n_0 > 0$ such that $0 \leq c_1 n^b \leq (n+a)^b \leq c_2 n^b$, $\forall\, n \geq n_0$. Note that

$$n + a \leq n + |a| \leq 2n \text{ when } |a| \leq n, \text{ and}$$
$$n + a \geq n - |a| \geq \frac{1}{2}n \text{ when } |a| \leq \frac{1}{2}n.$$

   Thus, when $n \geq 2|a|$, $0 \leq \frac{1}{2}n \leq n + a \leq 2n$. Since $b > 0$, the inequality still holds when all parts are raised to the power of b :

$$0 \leq \left(\frac{1}{2}n\right)^b \leq (n+a)^b \leq (2n)^b,$$
$$0 \leq \left(\frac{1}{2}\right)^b n^b \leq (n+a)^b \leq 2^b n^b.$$

   Thus, $c_1 = (1/2)^b$, $c_2 = 2^b$, and $n_0 = 2|a|$ satisfy the relation.

   (b) Prove or disprove the following statement $a^{2n} = O(a^n)$.

   Solution: If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists and is $\neq \infty$, then $f(n) = O(g(n))$. Now, we have $\lim_{n\to\infty} \frac{a^{2n}}{a^n} = \lim_{n\to\infty} \frac{a^n * a^n}{a^n} = \lim_{n\to\infty} a^n = \infty$. Hence, $a^{2n} \neq O(a^n)$.